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Carl B. Allendoerfer, *Editor*

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ON THE CONDITION OF RIEMANN INTEGRABILITY

HANS RADEMACHER, University of Pennsylvania

Some time ago Professor M. J. Norris published in this journal a paper* in which he proved the Riemann integrability of continuous functions without taking recourse to the notion of uniform continuity. Of course, the proof requires somewhere a transition from local properties to overall properties, and the author used for this purpose the mean value theorem of differential calculus. Since this theorem is easier for the beginner to grasp than the notion of uniform continuity, Professor Norris' proof is of great didactic merit. It turns out, however, that exactly the same proof exists already in the literature. It was given by Gerhard Kowalewski on pp. 174–176 of his little book *Grundzüge der Differential- und Integralrechnung* (Leipzig 1909).

When I read the Kowalewski-Norris proof for the Riemann integrability of continuous functions again it occurred to me that it can easily be widened so that it gives the Jordan theorem of the necessary and sufficient condition for Riemann integrability, and in a rather simple way, avoiding the Heine-Borel theorem and the Lebesgue integral. This proof I shall give in the following lines.

1. I have first to recall the definition of one-sided upper and lower derivatives. Let $f(x)$ be given as a real function in $a \leqq x \leqq b$, then we define the right upper derivative

$$\overline{D}_r f(x) = \overline{\lim_{x_1 \to x+0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

the right lower derivative

$$\underline{D}_r f(x) = \underline{\lim_{x_1 \to x+0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

and the left upper and the left lower derivatives respectively as

$$\overline{D}_l f(x) = \overline{\lim_{x_1 \to x-0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

$$\underline{D}_l f(x) = \underline{\lim_{x_1 \to x-0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

where $x_1 \to x+0$ is the customary symbol for ($x_1 > x$ and $x_1 \to x$). At a only the right derivatives are defined, at b only the left ones. We have, of course,

$$\overline{D}_r f(x) \geqq \underline{D}_r f(x), \qquad \overline{D}_l f(x) \geqq \underline{D}_l f(x).$$

Otherwise our definition does not exclude that any of the four derivatives is $+\infty$ or $-\infty$. A function is called differentiable at x_0 in the usual sense, if all

* Integrability of continuous functions, this MONTHLY vol. 59, 1952, pp. 244–245.

derivatives at x_0 are equal and finite.

With this definition we state a generalization of the theorem of the mean, given first by W. H. and G. C. Young in 1909.† We start with the generalization of Rolle's theorem, which we enunciate as

LEMMA 1. *If $f(x)$ is a continuous function in $a \leqq x \leqq b$, and if $f(a) = f(b)$, then there exists either a ξ_1, $a < \xi_1 < b$ such that*

$$\underline{D}_1 f(\xi_1) \geqq 0 \geqq \overline{D}_r f(\xi_1)$$

or there exists a ξ_2, $a < \xi_2 < b$, such that

$$\overline{D}_1 f(\xi_2) \leqq 0 \leqq \underline{D}_r f(\xi_2).$$

Proof: If for all x, $a < x < b$, we have $f(x) = f(a) = f(b)$, we may take $\xi_1 = \frac{1}{2}(a+b)$ and have $a < \xi_1 < b$ and

$$\underline{D}_1 f(\xi_1) = \overline{D}_r f(\xi_1) = f'(\xi_1) = 0.$$

Otherwise there will exist an x_0, $a < x_0 < b$, such that $f(x_0) \neq f(a)$. Let us assume $f(x_0) > f(a)$. Now $f(x)$ being continuous in a closed interval attains its maximum there, at ξ_1 say. Since $f(\xi_1) \geqq f(x_0) > f(a)$, $\xi_1 \neq a$ and $\xi_1 \neq b$, so that $a < \xi_1 < b$ follows. Now

$$\frac{f(x_1) - f(\xi_1)}{x_1 - \xi_1} \leqq 0 \qquad \text{for } x_1 > \xi_1$$

and thus

$$\overline{D}_r f(\xi_1) = \overline{\lim_{x_1 \to \xi_1 + 0}} \frac{f(x_1) - f(\xi_1)}{x_1 - \xi_1} \leqq 0,$$

and similarly

$$\underline{D}_1 f(\xi_1) \geqq 0.$$

In the case $f(x_0) < f(a)$ we take the minimum $f(\xi_2)$ and obtain the other alternative.

If we now lift the condition $f(a) = f(b)$ and consider in the well-known way the function

$$f_1(x) = f(x) - \frac{f(b) - f(a)}{b - a}\, x,$$

we obtain from Lemma 1 the generalized mean value theorem which we state as

LEMMA 2. *If $f(x)$ is a continuous function in $a \leqq x \leqq b$ then there exists either a ξ_1, $a < \xi_1 < b$, such that*

† On derivatives and the theorem of the mean, Quarterly Journal of Math. vol. 40, 1909, pp. 1–26, esp. p. 10, Theorem 4.

$$\underline{D}_1 f(\xi_1) \geqq \frac{f(b) - f(a)}{b - a} \geqq \overline{D}_r f(\xi_1),$$

or there exists a ξ_2, $a < \xi_2 < b$, *such that*

$$\overline{D}_1 f(\xi_2) \leqq \frac{f(b) - f(a)}{b - a} \leqq D_r f(\xi_2).$$

Corollary. If for the continuous function $f(x)$ *the lower derivatives* $\underline{D}_r f(\xi)$ *and* $\underline{D}_1 f(\xi)$ *in all interior points* ξ *have the property*

$$\underline{D}_r f(\xi) \leqq B, \qquad \underline{D}_1 f(\xi) \leqq B$$

then

$$\frac{f(b) - f(a)}{b - a} \leqq B.$$

2. Whereas the two lemmas dealt with continuous functions only we consider now real functions which need only to be bounded. Let $g(x)$ be defined in the closed interval $[a, b]$. We introduce the following definitions

$$M_g(\alpha, \beta) = \sup_{\alpha \leqq x \leqq \beta} g(x), \tag{2.1}$$

$$m_g(\alpha, \beta) = \inf_{\alpha \leqq x \leqq \beta} g(x), \tag{2.2}$$

and

$$\sigma_g(\alpha, \beta) = M_g(\alpha, \beta) - m_g(\alpha, \beta), \tag{2.3}$$

the "supremum," the "infinum," and the "oscillation" of $g(x)$ in the closed interval $[\alpha, \beta]$. (It is to be understood that if $[\alpha, \beta]$ should partly exceed the interval $[a, b]$ in which $g(x)$ is defined, the instruction "$\alpha \leqq x \leqq \beta$" in the definitions is to be applied only where meaningful, *i.e.* in the common part of $[a, b]$ and $[\alpha, \beta]$.) Since we assumed boundedness of $g(x)$, the numbers defined in (2.1), (2.2), (2.3) are finite.

For $h > 0$ the function of h

$$M_g(x_0 - h, x_0 + h)$$

is monotone non-decreasing since a supremum can only grow if the sample of which it is taken is enlarged. Similarly

$$m_g(x_0 - h, x_0 + h)$$

is monotone non-increasing, and

$$\sigma_g(x_0 - h, x_0 + h)$$

monotone non-decreasing as functions of $h > 0$.

Since these functions are bounded, the limits

$$\lim_{h\to+0} M_g(x_0 - h, x_0 + h) = M_g(x_0),$$

$$\lim_{h\to+0} m_g(x_0 - h, x_0 + h) = m_g(x_0),$$

$$\lim_{h\to+0} \sigma_g(x_0 - h, x_0 + h) = \sigma_g(x_0),$$

the "local supremum," "local infinum," "local oscillation" of g at x_0 exist. We remark that a function is continuous at x_0 if its local oscillation at x_0 is 0.

LEMMA 3. *$M_g(x)$ and $\sigma_g(x)$ are upper semi-continuous and $m_g(x)$ is lower semi-continuous.*

Proof: Upper semi-continuity of $M_g(x)$ at x_0 means:

$$\overline{\lim_{x_1\to x_0}} M_g(x_1) \leqq M_g(x_0).$$

Let $|x_1-x_0| < \delta$, and $0 < h < \delta - |x_1-x_0|$. Then the interval $[x_1-h, x_1+h]$ lies in the interior of the interval $[x_0-\delta, x_0+\delta]$, and therefore

$$M_g(x_1) \leqq M_g(x_1 - h, x_1 + h) \leqq M_g(x_0 - \delta, x_0 + \delta);$$

thus

$$\overline{\lim_{x_1\to x_0}} M_g(x_1) \leqq M_g(x_0 - \delta, x_0 + \delta).$$

This is true for any $\delta > 0$, and thus, for $\delta \to +0$

$$\overline{\lim_{x_1\to x_0}} M_g(x_1) \leqq M_g(x_0),$$

as we had to prove. The other statements of the lemma follow in a similar manner.

3. We come now to the main object of our discussion.

THEOREM 1. *A necessary condition for the Riemann integrability of the function $f(x)$ bounded in the interval $a \leqq x \leqq b$ is that, for every $\eta > 0$, the set S_η of all x for which $\sigma_f(x) \geqq \eta$ can be covered by finitely many intervals of total length at most η.*

Remark: If the condition of this theorem is fulfilled, then for a *fixed* $\eta > 0$ the set S_η can be covered by finitely many intervals of any total length $\epsilon < \eta$. Indeed, if $0 < \epsilon < \eta$ then by definition

$$S_\epsilon \supset S_\eta,$$

and thus since S_ϵ can be covered by a finite set of intervals of total length ϵ at most, S_η is in this way also covered.

Proof of THEOREM 1: Suppose the condition is not fulfilled. This would mean that there exists an exceptional $\eta_0 > 0$ such that any finite interval set covering S_{η_0} has a total length $\geqq \eta_0$. It is convenient to distinguish here two cases:

1) Suppose that S_{η_0} contains a whole interval I of length λ. We take now a partition $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ of the interval $[a, b]$ and form the difference of the upper and lower Darboux sums for that partition:

$$\overline{S}-\underline{S}=\sum_j M_f(x_{j-1}, x_j)(x_j-x_{j-1})-\sum_j m_f(x_{j-1}, x_j)(x_j-x_{j-1})$$
$$=\sum_j \sigma_f(x_{j-1}, x_j)(x_j-x_{j-1}).$$

Of this sum we consider only those summands which belong to intervals $[x_{j-1}, x_j]$ which have a point of I in the interior. Since such a point, say ξ, belongs to S_{η_0}, we can say for the interval $[x_{j-1}, x_j]$ surrounding it that

$$\sigma_f(x_{j-1}, x_f)\geqq\sigma_f(\xi)\geqq\eta_0.$$

On the other hand these intervals together cover I and therefore have a total length λ at least. Thus

$$\overline{S}-\underline{S}\geqq\eta_0\lambda$$

for Darboux sums belonging to any partition, hence the upper and lower Darboux integrals differ by at least $\eta_0\lambda$, and $f(x)$ cannot be integrable.

2) Suppose that S_{η_0} contains no interval, *i.e.*, does not possess an interior point. Assume that we have a partition of (a, b) of fineness $\frac{1}{2}\delta$, *i.e.*,

$$x_j-x_{j-1}\leqq\frac{\delta}{2}.$$

We can see to it that no point of the partition (except possibly a and b) belongs to S_{η_0}. If x_j should belong to S_{η_0}, we can find in a distance less than $\frac{1}{4}\delta$ another point x_j' not belonging to S_{η_0}. The so corrected partition $(a<x_1'<\cdots<x_{n-1}'<b)$ has still the fineness δ. Then each point of S_{η_0} (except possibly a and b) is contained in the *interior* of an interval of the partition. We consider only such intervals $[x_{j-1}', x_j']$ which have a point ξ of S_{η_0} in their interior. For these we have again

$$\sigma_f(x_{j-1}', x_j')\geqq\sigma_f(\xi)\geqq\eta_0.$$

Since these intervals must cover S_{η_0} their total length adds up at least to η_0, and we have

$$\overline{S}-\underline{S}=\sum_j \sigma(x_{j-1}', x_j')(x_j'-x_{j-1}')\geqq\eta_0^2$$

which again excludes Riemann integrability.

4. We come now to that part of our argument which resembles that one given by Kowalewski and Norris, and which leads to the converse of Theorem I.

THEOREM II. *A sufficient condition for Riemann integrability in $a\leqq x\leqq b$ of*

the bounded function $f(x)$ is that for every $\eta>0$ the S_η, defined as in THEOREM I, *can be covered by finitely many intervals of total length η at most.*

Proof: We consider the function

$$F(z) = \overline{\int_a^z} f(x)dx - \underline{\int_a^z} f(x)dx,$$

which certainly fulfills

$$F(z) \geqq 0.$$

We assume $|f(x)| \leqq M$, and have then as consequence of the well-known additivity of Darboux integrals for $h>0$

$$\begin{aligned} 0 \leqq F(z+h) - F(z) &= \overline{\int_a^{z+h}} f(x)dx - \overline{\int_a^z} f(x)dx - \underline{\int_a^{z+h}} f(x)dx + \underline{\int_a^z} f(x)dx \\ &= \overline{\int_z^{z+h}} f(x)dx - \underline{\int_z^{z+h}} f(x)dx \\ &\leqq Mh + Mh = 2Mh, \end{aligned}$$

and similarly for $h<0$

$$|F(z+h) - F(z)| \leqq 2M|h|,$$

which shows the continuity of $F(z)$.

As far as the derivatives are concerned we see first, for $h>0$,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\left\{\overline{\int_z^{z+h}} f(x)dx - \underline{\int_z^{z+h}} f(x)dx\right\} \\ &\leqq M_f(z, z+h) - m_f(z, z+h) = \sigma_f(z, z+h) \\ &\leqq \sigma_f(z-h, z+h) \end{aligned}$$

and therefore

$$\begin{aligned} \overline{D}_r F(z) &= \overline{\lim_{h\to+0}} \frac{1}{h}\{F(z+h) - F(z)\} \\ &\leqq \lim_{h\to+0} \sigma_f(z-h, z+h) = \sigma_f(z). \end{aligned}$$

In the same manner

$$\overline{D}_l F(z) \leqq \sigma_f(z)$$

follows. Altogether we have

$$\underline{D}_l F(z) \leqq \overline{D}_l F(z) \leqq \sigma_f(z), \quad . \tag{4.1}$$

$$\underline{D}F(z) \leqq \overline{D}F(z) \leqq \sigma_f(z). \tag{4.2}$$

It will be convenient to introduce a function of an interval $I=(\alpha, \beta)$, for $\alpha<\beta$, as follows:

$$F(I) = F(\alpha, \beta) = F(\beta) - F(\alpha).$$

Now let S_η be covered by finitely many intervals $I_j=(\alpha_j, \beta_j)$ of total length η at most. The complementary intervals we call $\overline{I}_j'$. We have then

$$F(b) = F(b) - F(a) = \sum_j F(I_j) + \sum_j F(I_j'). \tag{4.3}$$

In $\overline{I}_j'$ we have throughout

$$\sigma_f(x) < \eta$$

whereas in general $|f(x)| \leqq M$ entails

$$\sigma_f(x) \leqq 2M.$$

The inequalities (4.1) and (4.2) together with the Corollary show therefore

$$F(I_j) \leqq 2M \cdot l(I_j)$$

$$F(I_j') \leqq \eta \cdot l(I_j')$$

where $l(I)$ stands for the length of I. We infer therefore from (4.3)

$$F(b) \leqq 2M \sum_j l(I_j) + \eta \sum_j l(I_j')$$

$$\leqq 2M\eta + \eta(b - a) = \eta(2M + b - a).$$

Since η is any positive number and $F(b) \geqq 0$ we obtain now

$$F(b) = 0,$$

or explicitly,

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0,$$

which proves our theorem.

5. So far we have remained in the realm of the elementary theory of Riemann-Darboux integrals. But we can now easily connect our two theorems with the Lebesgue theory and prove

THEOREM III. *Necessary and sufficient condition for the Riemann integrability of the bounded function $f(x)$ is that the set S_0 of those x where $\sigma_f(x)>0$ is of Lebesgue measure zero.*

Proof: It is clear firstly that

$$S_0 = S_1 \cup S_{1/2} \cup S_{1/3} \cup \cdots \cup S_{1/n} \cup \cdots . \tag{5.1}$$

Now the *necessary* condition of Theorem I implies, in virtue of the "Remark" following it, that all S_η are of Lebesgue measure 0, and then (5.1) shows that

$$m_L S_0 = 0 \tag{5.2}$$

(m_L meaning Lebesgue measure).

Assume now conversely that is true. Then, since

$$S_\eta \subset S_0$$

we have also

$$m_L S_\eta = 0 \tag{5.3}$$

for any $\eta > 0$. This means that S_η can be covered by *denumerably* many intervals of at most, let us say, total length η.

But now, as we shall see in a moment, S_η is closed, and can therefore, in virtue of the Heine-Borel theorem, be covered with only *finitely* many among those infinitely many covering intervals, and thus (5.3) implies that S_η can be covered with *finitely* many intervals of total length η at most, which shows, according to Theorem II, that (5.3) is also *sufficient* for the Riemann integrability of $f(x)$.

The closure of S_η, comprising those points x in which $\sigma_f(x) \geqq \eta$, follows immediately from Lemma 3.

Of course, the arguments of section 5, employing the Heine-Borel theorem, which is essentially equivalent to the theorem of uniform continuity of a continuous function in a closed interval, overstep the limits which this note had set itself. Its main purpose remains the proof given for Theorem II.

THE EQUATION $X^2+PX+Q=0$ IN BINARY MATRICES

H. S. THURSTON and MARY K. ALEXANDER, University of Alabama

1. Introduction. In a series of papers, [3], [4], [5], published in 1884 Sylvester discussed the equation

$$f(X) = X^2 + PX + Q = 0 \tag{1}$$

where P and Q are either binary matrices or quaternions. His technique, presented here in modified form for the matric case, leads to a cubic equation $\phi(\lambda) = 0$ whose roots determine, in general, six solutions of (1). If, however, $\phi(0) = 0$, fewer solutions may exist. Sylvester called this the *irregular* case. While he treated the regular case in considerable detail, he disposed of the

irregular case with the remark that "some of the roots may disappear from the sphere of actuality, or may remain actual but become indeterminate, or these two states of affairs may coexist." The purpose of our paper is to throw some light on this seemingly chaotic situation.

2. The regular case. Any solution $X = X_1$ of (1) satisfies its own characteristic equation

$$x^2 + bx + d = 0. \tag{2}$$

Thus $X_1^2+BX_1+D=0$, where $B=bI$ and $D=dI$, and subtracting this from (1) we have

$$(P - B)X_1 + (Q - D) = 0.$$

Conversely, if $X=X_1$ with characteristic equation (2) is a solution of

$$(P - B)X + (Q - D) = 0, \tag{3}$$

it is also a solution of (1). The problem of solving (1) is, in effect, that of finding all pairs (b, d) which can exist for a potential solution of (3). If for any such pair $P-B$ is non-singular, then (3) has the unique solution $X=-(P-B)^{-1}(Q-D)$. On the other hand, if $P-B$ is singular, there may or may not exist a solution of (3) with characteristic equation (2). It will be shown that the singularity of $P-B$ is concomitant with the existence of a root $\lambda=0$ of $\phi(\lambda)=0$.

Since any scalar solution of (1) can be found by inspection we shall assume the existence of a non-scalar solution with characteristic equation (2). If

$$P = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \sigma & \xi \\ \eta & \zeta \end{bmatrix},$$

(3) can be written in the form

$$\begin{bmatrix} \alpha - b & \beta \\ \gamma & \delta - b \end{bmatrix} X + \begin{bmatrix} \sigma - d & \xi \\ \eta & \zeta - d \end{bmatrix} = 0. \tag{3'}$$

By a well-known theorem [2], any solution of (3') will satisfy

$$\begin{vmatrix} (\alpha - b)x + \sigma - d & \beta x + \xi \\ \gamma x + \eta & (\delta - b)x + \zeta - d \end{vmatrix} = 0. \tag{3''}$$

This takes the form $px^2+qx+r=0$, where

$$\begin{aligned} p &= b^2 - 2\tau_1 b + \Delta_1, \qquad q = 2bd - 2\tau_2 b - 2\tau_1 d + \alpha\zeta + \delta\sigma - \beta\eta - \gamma\xi, \\ r &= d^2 - 2\tau_2 d + \Delta_2, \end{aligned} \tag{4}$$

$2\tau_1$ and $2\tau_2$ being the traces of P and Q, Δ_1 and Δ_2 their determinants, respectively. The assumed solution of (1) being non-scalar, the equations $px^2+qx+r=0$ and $x^2+bx+d=0$ must be equivalent unless $p=q=r=0$, and corresponding coefficients are proportional. Even if p, q, and r are all zero we may still write $p=\lambda$, $q=b\lambda$, $r=d\lambda$. On solving the first and third of these we find respectively

$$b = \tau_1 \pm b_1, \qquad d = \tfrac{1}{2}(\lambda + 2\tau_2 \pm d_1) \tag{5}$$

where b_1 and d_1 are irrational. By observing the correspondence of sign (+ with + and − with −), on substituting these in $q=b\lambda$, the latter reduces to

$$b_1 d_1 = \tau_1\lambda + 2\tau_1\tau_2 - (\alpha\zeta + \delta\sigma - \beta\eta - \gamma\xi)$$

which is rationalized on squaring. We are thus led to a cubic equation $\phi(\lambda)=\lambda^3+u\lambda^2+v\lambda+w=0$, where

$$\begin{aligned} u &= 4\tau_2 - \Delta_1, \qquad v = 4\tau_2^2 - 4\Delta_2 - 4\Delta_1\tau_2 + 2\tau_1(\alpha\zeta + \delta\sigma - \beta\eta - \gamma\xi), \\ w &= \eta\xi(\alpha-\delta)^2 + \beta\gamma(\sigma-\zeta)^2 - (\beta\eta - \gamma\xi)^2 - (\alpha-\delta)(\sigma-\zeta)(\beta\eta+\gamma\xi). \end{aligned} \tag{6}$$

THEOREM 1. *The determinants of $P-B$ and $Q-D$ are respectively $p=\lambda$ and $r=d\lambda$.*

This is obvious from (3′) and (4).

THEOREM 2. *If $\lambda=0$ is not a root of $\phi(\lambda)=0$ then* (1) *has at most six non-scalar solutions.*

It is clear from (5) that corresponding to a root $\lambda_1\neq 0$ there are at most two pairs (b, d). Since for any such pair $P-B$ is non-singular, by Theorem 1, then (3) and hence (1) have a unique solution. If $\phi(\lambda)=0$ has no multiple roots, the six pairs (b, d) are distinct,* and if in addition there is no root $\lambda=0$, six solutions of (1) are obtained.

Example 1. If

$$f(X) = X^2 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} -7 & 1 \\ 0 & 0 \end{bmatrix} = 0,$$

$\phi(\lambda)$ is found to be $(\lambda-1)(\lambda-5)(\lambda-10)$. Corresponding to each triple (λ, b, d) we have a solution as follows:

$$(1, 1, -6), \quad X = \begin{bmatrix} 0 & 6 \\ 1 & -1 \end{bmatrix}; \qquad (1, 1, 0), \quad X = \begin{bmatrix} 0 & 0 \\ 7 & -1 \end{bmatrix};$$

$$(5, 3, 0), \quad X = 1/5\begin{bmatrix} -14 & 2 \\ 7 & -1 \end{bmatrix}; \quad (5, -1, -2) \quad X = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix};$$

$$(10, 4, 3), \quad X = \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix}; \qquad (10, -2, 0), \quad X = 1/10\begin{bmatrix} 21 & -3 \\ 7 & -1 \end{bmatrix}.$$

* From the fact, noted by Sylvester, that $\phi(\lambda)=0$ and $F(x)=0$ (defined in section 6) have the same discriminant.

3. The irregular case. If (b, d) is a pair corresponding to a root $\lambda=0$ we may assume a solution

$$X=\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

of (3). Then (3) becomes

$$\begin{bmatrix} \alpha-b & \beta \\ \gamma & \delta-b \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}+\begin{bmatrix} \sigma-d & \xi \\ \eta & \zeta-d \end{bmatrix}=0.$$

The solution X can be found if and only if the following system of six equations is consistent, in which case (b, d) is called an *admissible* pair.

$$\begin{aligned} (\alpha-b)x_1+\beta x_3+\sigma-d&=0\\ (\alpha-b)x_2+\beta x_4+\xi&=0\\ \gamma x_1+(\delta-b)x_3+\eta&=0\\ \gamma x_2+(\delta-b)x_4+\zeta-d&=0\\ x_1+x_4=-b,\qquad x_1x_4-x_2x_3&=d. \end{aligned} \tag{7}$$

The first four equations are consistent if and only if the matrix

$$N=\begin{bmatrix} \alpha-b & \beta & \sigma-d & \xi \\ \gamma & \delta-b & \eta & \zeta-d \end{bmatrix}$$

is of rank $\rho<2$.

If $\rho=0$, $P-B=Q-D=0$, and any matrix

$$X=\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

for which the last two equations of (7) are satisfied will be a two-parameter solution of (3) and hence of (1). This case arises when P and Q are both scalar matrices.

If, however, $\rho=1$, the system of four equations will reduce to two equations which with $x_1+x_4=-b$ may be solved for three of the x_i's (either uniquely or with the remaining x_i as a parameter) and the results substituted in the sixth equation as a further test of consistency. Which x_i may serve as parameter depends on which elements of $P-B$ are not zero. In any case it may be chosen as x_2 or x_3, the result of the substitution in the sixth equation of (7) being indicated as follows:

$\alpha-b\neq 0$: $[\xi(\alpha-b)-\beta(\sigma-d)]x_3=(\alpha-b)^2d-b(\alpha-b)(\sigma-d)+(\sigma-d)^2$

$\beta\neq 0$: $[-\xi(\alpha-b)+\beta(\sigma-d)]x_2=\beta^2d-b\beta\xi+\xi^2$

$\gamma\neq 0$: $[-\eta(\delta-b)+\gamma(\zeta-d)]x_3=\gamma^2d-b\gamma\eta+\eta^2$

$\delta-b\neq 0$: $[\eta(\delta-b)-\gamma(\zeta-d)]x_2=(\delta-b)^2d-b(\delta-b)(\zeta-d)+(\zeta-d)^2$.

If no element of $P-B$ is zero, any x_i may serve as parameter. For example, either x_1 or x_4 may be chosen, substitution in (7) yielding

$$(\beta\eta - \xi\gamma)x_1 = \beta\gamma d - b\beta\eta + \xi\eta, \tag{8}$$

or

$$(\xi\gamma - \beta\eta)x_4 = \beta\gamma d - b\gamma\xi + \xi\eta. \tag{8'}$$

In any of these cases if the coefficient of the x_i finally exhibited is not zero, we may solve uniquely for this unknown and a single solution of (3) is found. But if the coefficient is zero, there is a solution involving a parameter x_i, or no solution, according as the right member of the corresponding equation is or is not zero. We may therefore state the following theorem.

THEOREM 3. *If $\phi(\lambda)=0$ has a root $\lambda=0$, there may be fewer than six admissible pairs (b, d). To each admissible pair associated with $\lambda=0$ may correspond either a unique solution or a parametric solution of* (1).

On referring to (6) it is seen that, by definition, the vanishing of w is a necessary and sufficient condition for the occurrence of the irregular case. The form in which w is expressed suggests various sufficient conditions. For example, if P or Q is scalar, or if $\eta=\gamma=0$, w will vanish. If $\beta\eta=\gamma\xi$, w will reduce to $\eta[\xi(\alpha-\delta)-\beta(\sigma-\zeta)]^2$, so that $\xi(\alpha-\delta)=\beta(\sigma-\zeta)$ together with $\beta\eta=\gamma\xi$ is sufficient for the occurrence of the irregular case. These conditions hold if P is a linear function of Q, say $P=\mu Q+\nu$, since then $\beta=\mu\xi$, $\gamma=\mu\eta$, $\alpha-\delta=\mu(\sigma-\zeta)$. Thus we have

THEOREM 4. *A sufficient condition that equation* (1) *be irregular is that P be a linear function of Q.*

This case presents some aspects which were not apparent in the general discussion of this section and will now be investigated in detail.

4. The matric equation $f(X, Q)=0$. We shall now consider the equation

$$f(X, Q) = X^2 + (\mu Q + \nu)X + Q = 0 \tag{9}$$

where μ and ν are scalars, $\mu\neq 0$, and Q is a non-scalar matrix. With the trace of Q being denoted by 2τ and the determinant of Q by Δ, $\phi(\lambda)$ is of the form $\lambda^3+k\lambda^2+4mn\lambda$ where

$$k = 4\tau - \mu^2\Delta - 2\mu\nu\tau - \nu^2, \qquad m = 1 - \mu\nu, \qquad n = \tau^2 - \Delta. \tag{10}$$

For $\lambda=0$ the coefficients b and d given by (5) become $b=\mu d+\nu$, $d=\tau\pm\theta$, where $\theta^2=n$. Equation (3') takes the form

$$\mu(Q - D)X + Q - D = 0. \tag{11}$$

Thus the rank of N is clearly less than two, by Theorem 1 and (11), and the first four equations of the system (7) are consistent. The final substitution in

the sixth equation of the system leads to a consistent result if and only if $m=0$. For example, in (8), since $\beta\eta-\gamma\xi=0$, the consistency depends upon the vanishing of the right member. But

$$\begin{aligned}\beta\gamma d-\beta\eta b+\xi\eta &= \mu\xi[\mu\eta d-\eta(\mu d+\nu)]+\xi\eta\\ &= -\mu\nu\xi\eta+\xi\eta]\\ &= \xi\eta(1-\mu\nu)\\ &= 0\end{aligned}$$

if and only if $m=0$, ξ and η being in this case different from zero. The same condition can be shown to be necessary and sufficient for a solution in each of the cases that can arise. Thus we have proved

THEOREM 5. *A necessary and sufficient condition that there be an admissible pair* (b, d) *associated with* $\lambda=0$, *and hence a corresponding solution of* (9), *is that* $m=0$.

As pointed out earlier, there will in this case be a parametric solution, the choice of parameter being determined by the non-zero elements of $P-B$. In any case x_2 or x_3 may be chosen as parameter to avoid zero denominators. However, we shall take the case where there are no zero elements as typical, and since the conclusions are the same for all cases, no further reference will be made to the others. We can now express x_2, x_3, and x_4 as follows:

$$x_2=\frac{-\xi(x_1+\mu d)}{d-\sigma},\qquad x_3=\frac{\eta(x_1+\nu)}{d-\zeta},\qquad x_4=-(x_1+\nu+\mu d),$$

where neither of the denominators is zero. Since $d=\tau\pm\theta$ is double-valued (unless $n=0$), there are two parametric solutions in x_1 corresponding to a pair (b, d). If $m=n=0$, only one parametric solution is obtained. We have proved

THEOREM 6. *When* $m=0$, *corresponding to* $\lambda=0$ *there are two parametric solutions if* $n\neq 0$, *one such solution if* $n=0$.

If $k=m=n=0$, then $\mu\tau=\nu$, and the single parametric solution is

$$X=\begin{bmatrix} x_1 & \dfrac{\xi(x_1+\nu)}{\sigma-\tau}\\[2ex] \dfrac{\eta(x_1+\nu)}{\tau-\zeta} & -(x_1+2\nu)\end{bmatrix}. \tag{12}$$

When $m=0$ it is clear that

$$f(X, Q)=X^2+(\mu Q+\nu)X+\mu\nu Q=(X+\nu)(X+\mu Q)$$

and $X=-\nu I$, $X=-\mu Q$ are solutions of (9). In case $k\neq 0$ these solutions correspond to $\lambda=-k$. However, if $k=0$, they become particular solutions with the

parameter x_1 taking on the values $-\nu$ and $-\mu\sigma$ respectively. This is obvious from (12) when $n=0$ and easily verified when $n\neq 0$.

Example 2. Let $f(X, Q)=X^2+(Q+1)X+Q=0$ where

$$Q=\begin{bmatrix}3 & -1\\ 0 & 2\end{bmatrix}.$$

Here $m=0$, $k\neq 0$, $n\neq 0$ and $\phi(\lambda)=\lambda^2(\lambda-2)$. Corresponding to $\lambda=2$ are the two solutions $X=-I$, $X=-Q$, while $\lambda=0$ yields the parametric solutions

$$X=\begin{bmatrix}-3 & x_2\\ 0 & -1\end{bmatrix}, \qquad X=\begin{bmatrix}x_1 & -x_1-2\\ x_1+1 & -x_1-3\end{bmatrix}.$$

Example 3. $f(X, Q)=X^2+(\frac{1}{2}Q+2I)X+Q=0$ where

$$Q=\begin{bmatrix}6 & 2\\ -2 & 2\end{bmatrix}.$$

We now have $m=n=k=0$ and $\phi(\lambda)=\lambda^3$. One parametric solution

$$X=\begin{bmatrix}x_1 & x_1+2\\ -x_1-2 & -x_1-4\end{bmatrix}$$

is obtained yielding the particular solutions $X=-2I$, $X=-\frac{1}{2}Q$ when $x_1=-2$ and -3 respectively.

5. Other special cases.

THEOREM 7. *If P is scalar and Q non-scalar there is no solution of* (1) *corresponding to $\lambda=0$; if Q is scalar and P non-scalar there is a solution corresponding to $\lambda=0$ if and only if $Q=0$.*

Under the first hypothesis $P-B=0$, $Q-D\neq 0$, and obviously there is no solution of (3). This is the case $P=\mu Q+\nu$, where $\mu=0$.

If $Q=\sigma I$ and P is non-scalar, the six equations (7) are found to be consistent if and only if $\sigma=0$, in which case parametric solutions are obtained.

The final special case to consider is that in which Q is scalar and $P=\mu Q+\nu$ is also scalar. If $Q=\sigma I$, Sylvester's method leads to $\phi(\lambda)=\lambda^2(\lambda+k)$ where $-k$ is the discriminant of $f(x, \sigma)=0$. To the root $\lambda=-k$ there correspond the scalar solutions $X_1=r_1I$, $X_2=r_2I$ where r_1 and r_2 are the roots of $f(x, \sigma)=0$. As noted earlier, for $\lambda=0$ the rank of N is zero and there is a two-parameter solution subject to the conditions $x_1+x_4=-\mu\sigma-\nu$, $x_1x_4-x_2x_3=\sigma$. It is known that a polynomial equation with scalar coefficients has an infinite number of matric solutions of any desired order. Hermann [1] and Thurston [6], [7], have discussed the number of solutions in the ring $R(A)$ of any given matrix A. As in the following example, any such binary solution may be identified with one of the above-mentioned scalar or parametric solutions.

Example 4. Consider the equation $X^2+3IX+2I=0$. If

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

then solutions in $R(A)$ are given by $X=sA+t$ where $s=0, t=-1$; $s=0, t=-2$; $s=-1/3, t=-4/3$; $s=1/3, t=-5/3$. These yield respectively the solutions

$$X_1 = -I, \quad X_2 = -2I, \quad X_3 = -1/3\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}, \quad X_4 = -1/3\begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}.$$

The last two are easily identified as particular cases of the parametric solutions indicated above.

6. The associated quartic equation. The theorem [2] which permitted passing from (3′) to (3″) may be applied to (1) yielding the equation

$$F(x) = \begin{vmatrix} x^2+\alpha x+\sigma & \beta x+\xi \\ \gamma x+\eta & x^2+\delta x+\zeta \end{vmatrix} = 0,$$

or

$$\begin{aligned} F(x) &= x^4 + 2\tau_1 x^3 + (2\tau_2 + \Delta_1)x^2 \\ &\quad + (\sigma\delta + \alpha\zeta - \beta\eta - \gamma\xi)x + \Delta_2 = 0, \end{aligned} \tag{13}$$

and any solution of (1) satisfies (13).

THEOREM 8. *If $X=X_1$ is a solution of* (1) *with characteristic equation* $g(x)=x^2+bx+d=0$, *then* $g(x)$ *divides* $F(x)$.

Since $f(X_1)=X_1^2+PX_1+Q=0$, we may write

$$f(xI) = f(xI) - f(X_1) = x^2I^2 - X_1^2 + P(xI - X_1) = (xI + X_1 + P)(xI - X_1).$$

Then the determinant of $xI-X_1$ divides that of $f(xI)$. But the determinant of $xI-X_1$ is $g(x)$ and that of $f(xI)$ is $F(x)$.*

This theorem suggests an alternate method of finding pairs (b, d) for potential solutions† of (1). In the regular case the six possible quadratic factors of $F(x)$ are the characteristic functions of the six solutions mentioned in the introduction.

If $P=\mu Q+\nu$, the associated quartic may be written in the factored form $F(x)=F_1(x)F_2(x)$ where

$$\begin{aligned} F_1(x) &= x^2 + (\mu\tau + \nu + \mu\theta)x + (\tau + \theta), \\ F_2(x) &= x^2 + (\mu\tau + \nu - \mu\theta)x + (\tau - \theta). \end{aligned}$$

* This proof is essentially that given by Sylvester [5].

† This theorem also permits the omission of the word "non-scalar" from the enunciation of Theorem 2 since the maximum number of quadratic factors of $F(x)$ is six.

If $m=0$, $\mu\nu=1$, and we may write

$$F_1(x) = x^2 + (\mu\tau + \nu + \mu\theta)x + \mu\nu(\tau + \theta) = (x + \nu)(x + \mu\tau + \mu\theta).$$

Similarly,

$$F_2(x) = (x + \nu)(x + \mu\tau - \mu\theta),$$

so that

$$F(x) = (x + \nu)^2[(x + \mu\tau)^2 - (\mu\theta)^2] = (x + \nu)^2(x^2 + 2\mu\tau x + \mu^2\Delta). \tag{14}$$

If, in addition, $k=0$, then

$$k = 4\tau - \mu^2\Delta - 2\mu\nu\tau - \nu^2 = 2\tau - \mu^2\Delta - \nu^2 = 0,$$

whence

$$2\mu\tau = \mu^3\Delta + \mu\nu\cdot\nu = \mu^3\Delta + \nu.$$

Then

$$x^2 + 2\mu\tau x + \mu^2\Delta = x^2 + (\mu^3\Delta + \nu)x + \mu^3\nu\Delta = (x + \nu)(x + \mu^3\Delta).$$

Thus for $m=k=0$,

$$F(x) = (x + \nu)^3(x + \mu^3\Delta). \tag{15}$$

On the other hand if $m=0$ and $n=\theta^2=0$, we will have from (14),

$$\begin{aligned} F(x) &= (x + \nu)^2(x + \mu\tau)^2 \\ &= [x^2 + (\mu\tau + \nu)x + \mu\nu\tau]^2 \\ &= [x^2 + (\mu\tau + \nu)x + \tau]^2 \\ &= [f(x, \tau)]^2. \end{aligned} \tag{16}$$

Finally, if $m=n=k=0$, since $\mu\tau=\nu$, $\tau=\nu^2$, we have

$$F(x) = (x + \nu)^4. \tag{17}$$

The following table provides a complete summary of the results of this section as well as those of section 4. It is understood that $m=0$ in all four cases tabulated and the solutions indicated are only those corresponding to $\lambda=0$.

Cases	$\phi(\lambda)$	$F(x)$	Solutions
$k=0$, $n=0$	λ^2	$(x+\nu)^4$	one parametric
$k=0$, $n\neq0$	λ^2	$(x+\nu)^3(x+\mu^3\Delta)$	two parametric
$k\neq0$, $n=0$	$\lambda^2(\lambda+k)$	$[f(x, \tau)]^2$	one parametric
$k\neq0$, $n\neq0$	$\lambda^2(\lambda+k)$	$(x+\nu)^2(x^2+2\mu\tau+\mu^2\Delta)$	two parametric

7. Conclusion. We would not wish the readers of this paper to draw from the discussions of sections 4 and 6 the erroneous conclusion that all solutions corresponding to $\lambda=0$ are parametric. Neither would we wish them to conclude from the examples of these sections that there are solutions corresponding to $\lambda=0$ only when the latter is a multiple root of $\phi(\lambda)=0$. This is equally erroneous, as our final example demonstrates.

Example 5. If

$$f(X) = X^2 + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix},$$

we find that $\phi(\lambda)=\lambda^2$. Corresponding to $\lambda=0$ there is an inadmissible pair $b=2$, $d=1$, and an admissible pair $b=0$, $d=-1$, yielding a solution

$$X = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Example 6. Let

$$f(X) = X^2 + \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} X + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The root $\lambda=0$ of $\phi(\lambda)=\lambda(\lambda-2)^2=0$ leads to $b=2$, $d=1$, yielding no solution, and $b=-2$, $d=-1$, with the corresponding solution

$$X = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

In conclusion, the authors wish to express their appreciation for the suggestions and constructive criticisms of the referee in the preparation of this paper.

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A NOTE ON PRIMITIVE MATRICES†

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Suppose that A is a square matrix consisting of non-negative elements. In certain considerations it is important to know when all the elements of some power of A are strictly positive. Frobenius [2] gave a very simple necessary and sufficient condition for this to happen. In this note we give a simple proof of this result. Our proof is algebraic in nature and avoids the use of the convergence of powers of a matrix.

All matrices considered here will have real elements. For two such matrices (not necessarily square) $B=(b_{ij})$, $C=(c_{ij})$ we define

$$B \geqq C \qquad \text{if } b_{ij} \geqq c_{ij} \text{ for each } i, j,$$

$$B > C \qquad \text{if } b_{ij} > c_{ij} \text{ for each } i, j.$$

A square matrix $A \geqq 0$ (A is then called non-negative) is said to be *indecomposable* if there is no permutation matrix P for which

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where the A_{ii} are square submatrices

The fundamental result about non-negative, indecomposable matrices is due to Frobenius [2]; this, and other, results have recently been rederived and extended in a greatly simplified manner by Wielandt [3] and Debreu and Herstein [1]. It is the

THEOREM I. *Let $A \geqq 0$ be an indecomposable matrix. Then A has a positive characteristic root r such that*

(1) *r is a simple root;*

(2) *to r can be associated a characteristic vector $x > 0$;*

(3) *if α is any other characteristic root of A, $|\alpha| \leqq r$.*

If $A > 0$ then (3) can be sharpened to $|\alpha| < r$ for all characteristic roots $\alpha \neq r$ of A.

If $A \geqq 0$ is indecomposable and if A has no characteristic root other than r of maximal absolute value then A is said to be *primitive.*

In this paper we prove the

THEOREM II (Frobenius). *Let $A \geqq 0$. Then $A^m > 0$ for some integer $m > 0$ if and only if A is primitive.*

Suppose that $A^m > 0$. Then A must be indecomposable; for if

† This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under subcontract to the RAND Corporation.

$$PAP^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

then

$$PA^mP^{-1} = \begin{pmatrix} B^m & C^m \\ 0 & D^m \end{pmatrix}, \qquad \text{contradicting } A^m > 0.$$

Now suppose that r and $re^{i\phi} \neq r$ are characteristic roots of A of maximal absolute value. Then A^m, A^{m+1} are both positive and have r^m, $r^m e^{im\phi}$, and r^{m+1}, $r^{m+1}e^{i(m+1)\phi}$ respectively as roots of maximal absolute value. Since the largest root of a positive matrix is simple and is actually greater than any other root in absolute value we must have $r^m e^{im\phi} = r^m$, $r^{m+1}e^{i(m+1)\phi} = r^{m+1}$, whence $e^{i\phi} = 1$, a contradiction.

There remains but to show that if A is primitive then $A^m > 0$ for a suitable integer $m > 0$. This will be proved as a consequence of the following few lemmas, which by themselves are of some interest.

LEMMA 1. *If A is primitive then A^m is primitive for every positive integer m.*

Proof. Since r is a simple root of A and is the only root of A of absolute value r, r^m is a simple root of A^m and is the only root of A^m of absolute value r^m. So we need but show that A^m is indecomposable for every integer $m > 0$. Suppose that for some s, A^s is not indecomposable; we can then assume that

$$A^s = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

Now $Ax = rx$ for $x > 0$, so $A^s x = r^s x$; partition x according to the partitioning of A^s and we have

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

That is, $Dx_2 = r^s x_2$, and since x_2 is positive, r^s is a characteristic root of D. Since the transpose, A', of A is also indecomposable, we have $A'y = ry$ for $y > 0$. Partitioning as above we obtain that r^s is a characteristic root of B', and so of B. Being a characteristic root of both B and D, r^s must be a multiple root of A^s, which is a contradiction. The lemma is thereby proved.

LEMMA 2. (Wielandt). *Let ϵ be any positive number. Suppose $A \geqq 0$ is an $n \times n$ indecomposable matrix. Then $(\epsilon I + A)^{n-1} > 0$ where I is the identity matrix.*

Proof. It clearly suffices to show that for any vector x, $x \geqq 0$, $(\epsilon I + A)^{n-1}x > 0$. Let

$$x_r = (\epsilon I + A)^{r-1}x.$$

Then

$$x_{\nu+1} = \epsilon x_\nu + A x_\nu.$$

Hence a zero component can occur in $x_{\nu+1}$ only where a zero component already occurred in x_ν. However, not every such zero component can be preserved in $x_{\nu+1}$. For if so, by a suitable reordering of the coordinates,

$$x_\nu = \begin{pmatrix} p \\ 0 \end{pmatrix}, \qquad p > 0,$$

whence

$$x_{\nu+1} = \epsilon \begin{pmatrix} p \\ 0 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

from which it follows that $A_{21}p = 0$. This together with $p > 0$ forces $A_{21} = 0$, violating the indecomposability of A. So each application of $\epsilon I + A$ to x decreases the number of zero coordinates by at least one. Hence $(\epsilon I + A)^{n-1}x > 0$.

As an easy consequence of Lemma 2 we obtain

LEMMA 3. *If $A = (a_{ij})$ is indecomposable and $a_{ii} > 0$ for each i then $A^{n-1} > 0$.*

For let ϵ be chosen satisfying $0 < \epsilon < \min_i a_{ii}$. Then $A = \epsilon I + B$ where $B \geqq 0$ is indecomposable. Lemma 2 then yields $A^{n-1} > 0$.

Let $A^m = (a_{ij}^{(m)})$. Then we have

LEMMA 4. *Let $A \geqq 0$ be indecomposable. Then for any i, j we can find an $m = m(i, j) > 0$ so that $a_{ij}^{(m)} > 0$.*

Proof. Consider first the case $i \neq j$. Since

$$(I + A)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \cdots + I > 0$$

by Lemma 2, $a_{ij}^{(m)} > 0$ for some $m \leqq n-1$. Now suppose $i = j$. Since A is indecomposable, no column of zeros can occur in A. So there is a k with $a_{ki} > 0$. If $k = i$ then $a_{ii}^{(m)} > 0$ for all m trivially. If, on the other hand, $k \neq i$, then $a_{ik}^{(m)} > 0$ for some m, and since $a_{(m+1)}^{(m)} = \sum_r a_{ir}^{(m)} a_{ri} \geqq a_{ik}^{(m)} a_{ki} > 0$, the lemma is proved.

We are now in position to complete the proof of Theorem II. Let A be primitive. Pick m_1 so that in A^{m_1}, $a_{11}^{(m_1)} > 0$. Let $A_1 = A^{m_1} = (a_{ij}(1))$. By Lemma 1, A_1 is primitive, so there is an m_2 such that in $A_1^{m_2}$, $a_{22}^{(m_2)}(1) > 0$. Since $a_{11}(1) = a_{11}^{(m_1)} > 0$, $a_{11}^{(m_2)}(1) > 0$. Let $A_2 = A_1^{m_2}$. Continuing in this way we arrive at an $A_n = A^{m_1 m_2 \cdots m_n}$ which is primitive and whose diagonal elements are all positive. By Lemma 3, $A_n^i > 0$ for some i, hence $A^m > 0$ for some suitably chosen integer m.

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BOUNDED MODELS OF THE EUCLIDEAN PLANE

Editorial Note. Although the process of graphing $y=f(x)$ in the Euclidean plane is extremely well-known, comparatively little attention has been paid to the deficiencies of this method. The chief objection is that such graphs rarely illustrate the behavior of the function at large values of x and y. Quite independently several authors have submitted papers to the MONTHLY advocating the advantages of graphing such functions in a bounded model of the Euclidean plane. Graphs of this sort show the complete behavior of the function, but naturally involve distortions of the customary graph. As treated in detail below, R. L. Swain has proposed mapping the plane onto the interior of a square, and David Gans has discussed its map onto the interior of a circle. Gans' model has the further advantage of introducing the ideal points needed to convert the Euclidean plane into the plane of projective or elliptic geometry. It also shows the topological properties of these planes. Kenneth May has elaborated on Gans' idea by constructing special graph paper. In order to save space through the elimination of overlapping material, these three papers have been combined into the article below.

I. CONDENSED GRAPHS*

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1. The condensed plane. We map the Euclidean xy-plane onto the interior of the square bounded by the lines $x'=\pm 1$, $y'=\pm 1$ by the relations:

$$x'=\frac{x}{1+|x|}, \qquad y'=\frac{y}{1+|y|}. \tag{1}$$

Each point of the square region is now regarded as having the coordinates of that point of the original xy-plane which was mapped into it under (1). Each point of the boundary of the square region is also regarded as having coordinates, at least one of which is $+\infty$ or $-\infty$, assigned in the obvious way. The resulting closed square region is called the (rectangular) *condensed plane.*

It will be convenient to regard such a transformation as effected through the aid of a *mapping function* $\phi(t)$, defined for $0\leqq t<+\infty$, in this case the function

$$\phi(t)=\frac{t}{1+t}. \tag{2}$$

The relations in (1) are given by $x'=\phi(x)$ for $x\geqq 0$, $x'=-\phi(-x)$ for $x\leqq 0$, $y'=\phi(y)$ for $y\geqq 0$, $y'=-\phi(-y)$ for $y\leqq 0$. The essential features of the new scales given by the mapping will therefore be apparent if we discuss its effect in transforming just the non-negative portion of the x-axis.

Under the mapping, the non-negative portion of the original x-axis has been thrown into what was its unit (half-open) interval (0, 1). Each point x' of this interval is now "marked" with a new coordinate x, where $x'=\phi(x)$, that is, $x=x'/(1-x')$. The right end-point, which had the coordinate $x'=1$, is now marked with the symbol $+\infty$. This new scale is shown in Figure 1. The points to be marked (2^n-1) and $1/(2^n-1)$, *i.e.*, 1, 3, 7, 15, $\cdots$, 1/3, 1/7, 1/15, $\cdots$,

* This material was presented at the International Congress of Mathematicians, Cambridge, Massachusetts, 1950, *Proceedings Int. Congr. Math.*, vol. I, page 760, 1952.

I wish to thank the editor of the MONTHLY for helpful editorial suggestions.

Fig. 1

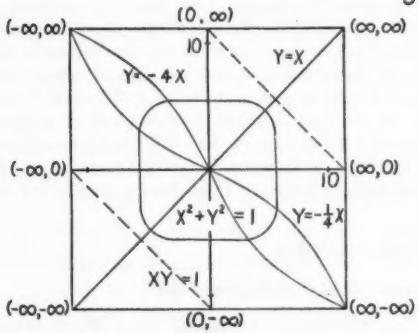

Fig. 2

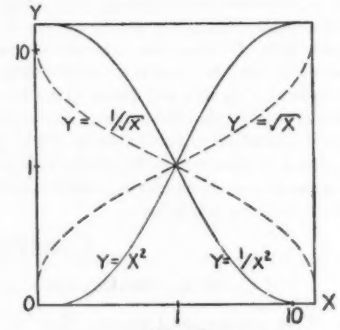

Fig. 3

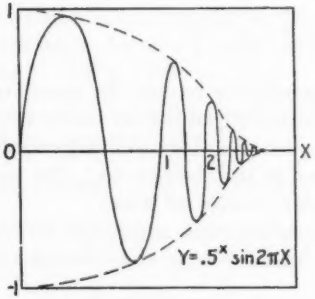

Fig. 4

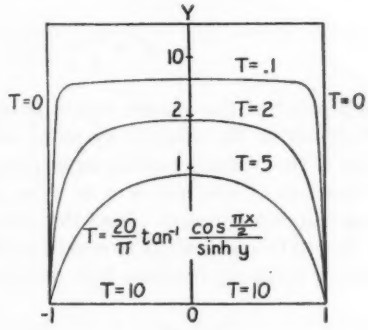

Fig. 5

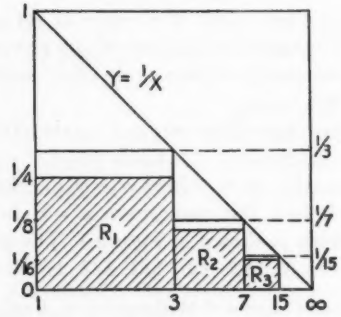

Fig. 6

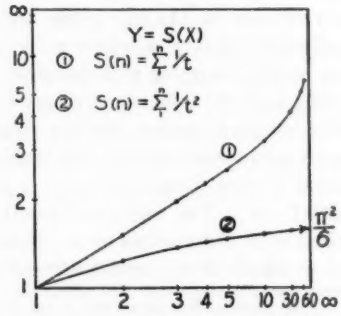

Fig. 7

may be located quickly and accurately in blackboard sketching because these are the "halfway" division points to the right and left from the midpoint, now marked 1, of the interval.

Given any equation $f(x, y)=0$, we may draw its complete graph in the condensed plane directly, by using the scale values marked on the axes. The graph of the line $y=x$ is the straight-line segment joining the points $(-\infty, -\infty)$ and $(+\infty, +\infty)$. The graph of the hyperbola $xy=1$ consists of two straight-line segments, one joining $(0, +\infty)$ and $(+\infty, 0)$, the other joining $(-\infty, 0)$ and $(0, -\infty)$. Besides these, Figure 2 shows the graphs of $x^2+y^2=1$, $y=-4x$ and $y=-(1/4)x$.

2. Drawn slope. The proportionate amount of scale condensation at the point marked x on the condensed x-axis is given by

$$dx' = \phi'(|x|)dx = \frac{1}{(1+|x|)^2}dx. \tag{3}$$

The derivative dy/dx will still be called the "slope" of the graph of an equation $f(x, y)=0$. But the graph in the condensed plane appears to have the slope dy'/dx'. This we call the *drawn slope* of the curve. From (3) we find

$$\frac{dy'}{dx'} = \left(\frac{1+|x|}{1+|y|}\right)^2 \frac{dy}{dx}. \tag{4}$$

Since dy'/dx' and dy/dx always have the same sign, the location of maxima and minima (relative to the marked scale) is not affected by graphing in the condensed plane. However, points of inflection are not preserved. The curve crosses the lines $y=\pm x$ with drawn slope equal to its slope.

The graph of every equation $y=mx+b$ $(m>0)$ terminates at $(+\infty, +\infty)$. We find the drawn slope at this point from (4):

$$\lim_{x\to\infty} \frac{dy'}{dx'} = m \lim_{x\to\infty} \left(\frac{1+x}{1+b+mx}\right)^2 = \frac{1}{m}. \tag{5}$$

3. Symmetries. In the ordinary plane, changing the sign of x or y or interchanging x and y results in a new graph symmetric with respect to the old. In the condensed plane, in addition to these usual symmetries, a new type appears, from taking the reciprocal of either x or y. The graphs of

$f(x)$ and $f(1/x)$ are symmetric w.r.t. the lines $\begin{cases} x=+1 & (x \geqq 0) \\ x=-1 & (x \leqq 0), \end{cases}$

$f(x)$ and $1/f(x)$ are symmetric w.r.t. the lines $\begin{cases} y=+1 & (f(x) \geqq 0) \\ y=-1 & (f(x) \leqq 0). \end{cases}$

These properties provide a useful teaching feature and may have some practical utility. They arise because the mapping function satisfies the identity

$$\phi(t) + \phi(1/t) = 1. \tag{6}$$

The $f(x)$ and $f(1/x)$ symmetry is established by taking $t=|x|$, the other by $t=|y|$. Some of these symmetries show up in Figures 2 and 3.

4. Asymptotic behavior. The asymptotic properties of a function are of special interest to the technical and engineering student as well as to the mathematician. Figure 4 shows the die-out with time x of the displacement in a damped harmonic oscillation. Figure 5 shows contour curves for a function, the isotherms in a semi-infinite strip with insulated faces whose three edges are maintained at 0°, 10°, and 0°, as marked.

5. Improper integrals. Suppose we are concerned with the convergence of an integral $\int_a^\infty f(x)dx$, where $f(x)$ is continuous and positive for $a<x<\infty$. Suppose that $f(x)$ is decreasing and that $\lim_{x\to\infty} f(x)=0$. In the condensed plane the graph of $y=f(x)$ then terminates at $(+\infty, 0)$. At this point the drawn slope of the curve $xy=k$ $(k>0)$ is $-k$. Consequently, if the drawn slope of the graph of $y=f(x)$ is negative at $(+\infty, 0)$, there will be some k value such that for sufficiently large x, the graph of $y=f(x)$ lies above the graph of $y=k/x$. In this case the integral $\int_a^\infty f(x)dx$ must diverge. It is therefore a necessary (but not a sufficient!) condition for convergence that the graph of $y=f(x)$ be tangent to the axis at $(+\infty, 0)$. Similarly, if $f(0+)=\infty$, $\int_0^b f(x)dx$ cannot converge unless the graph of $f(x)$ is tangent to the y-axis. The form of the graph of $y=1/x^2$, shown in Figure 3, makes it apparent that $\int_0^1 (1/x^2)dx$ diverges.

The divergence of the integral $\int_1^\infty (1/x)dx$ may be displayed graphically to the student by blocking in rectangles R_n of (scale) base length 2^n and of (scale) height $1/2^{n+1}$. Each has "true" area $1/2$. R_1 has the x-axis interval $(1, 3)$ for its base, R_2 the interval $(3, 7)$, *etc.* These rectangles lie within the drawn "squares" S_n with the same bases, which inscribe beneath the "line" $xy=1$. This is illustrated in Figure 6.

6. Series. The convergence or divergence of a series may be convincingly portrayed by plotting the partial sums $S(n)$ in the condensed plane. Figure 7 shows such plots for the series $\sum_1^\infty 1/t$ and $\sum_1^\infty 1/t^2$. (The points are joined by a smooth curve as a visual aid.)

It is also feasible to plot the test ratios $r(n)=u_{n+1}/u_n$ for a given series $\sum_1^\infty u_t$, or to plot the curve $r(x)=f(x+1)/f(x)$ in the case of an improper integral of form $\int_a^\infty f(t)dt$. If $|r(n)|$ approaches a limit other than unity, the graph will show this. In case $|r(n)|\to 1$, the following sufficient test, a graphical equivalent of Gauss' test, may aid: Given $\sum_1^\infty u_t$, a series of positive terms, plot the points $P_n=(n, r(n))$ in the condensed plane. If there exists a line l through $(+\infty, 1)$ of drawn slope exceeding $1/4$, such that for some N and all $n>N$, P_n lies below l, then the series converges. (A similar test may be made in the ordinary plane by plotting $P_n=(1/n, 1-r(n))$ and requiring P_n for $n>N$ to lie above a line $y=kx$ with $k>1$.)

7. Mapping functions. We selected $\phi(t)=t/(1+t)$ as our mapping function primarily for its simplicity. Many other choices are possible, such as $\phi(t)=1-e^{-t}$, $\phi(t)=(2/\pi)\tan^{-1}t$, *etc.** The following would appear to be minimum requirements to be met by a mapping function $\phi(t)$: *$\phi(t)$ should be a single-valued, twice differentiable, increasing function, defined for $0\leqq t<+\infty$, with $\phi''(t)<0$ for $t\neq 0$, $\phi(0)=0$, $\phi(+\infty)=1$.*

Given such a mapping function $\phi(t)$, the x-axis (or any distance coordinate) is subjected to the transformation $x'=\phi(x)$ for $x\geqq 0$, $x'=-\phi(-x)$ for $x\leqq 0$. This maps it onto the open interval $(-1, 1)$. Each point x' of this interval is now considered to have the coordinate x in the new scale, and the end-points are given new "coordinates" $\pm\infty$. Since $dx'=\phi'(|x|)dx$, $\phi'(|x|)$ is the measure of scale condensation at the point marked x on the (new) x-axis. Since $\phi''(t)<0$ for $t\neq 0$, $\phi'(|x|)$ decreases smoothly as $|x|$ increases.

A rectangular condensed plane is obtained by shrinking both x- and y-axes according to the same $\phi(t)$. The plane is thus mapped onto the interior of a square of side 2, with center at the origin. Horizontal lines are carried into horizontal lines, vertical into vertical, the origin being the only fixed point. The interior points of the square with original coordinates (x', y') are regarded as having the coordinates (x, y) in the new scale; the boundary of the square is appropriately scaled, at least one coordinate of each of its points being $-\infty$ or $+\infty$, and is adjoined to the interior. The resulting closed region is the rectangular condensed plane generated by the mapping function $\phi(t)$.

We have

$$\frac{dy'}{dx'}=\frac{\phi'(|y|)}{\phi'(|x|)}\frac{dy}{dx}. \tag{7}$$

This formula, the generalization of (4), gives the drawn slope of a curve at the point whose marked coordinates in the condensed plane are (x, y). Since $\phi'>0$, the drawn slope has the same sign as the "true" slope, so that the location of maxima and minima is not affected. The continuity of ϕ' insures that the drawn slope approximates the slope in the neighborhood of the origin. Curves cross the lines $y=\pm x$ with their drawn slope equal to their slope. The graph of $y=x$ is the straight-line segment joining $(-\infty, -\infty)$ to $(+\infty, +\infty)$.

In case the function $g(t)=\phi(1/t)$ has a Maclaurin expansion, it may be shown quite easily that the drawn slope of the line $y=mx+b$ $(m>0)$ at $(+\infty, +\infty)$ is $1/m^p$, where p is the exponent of t in the first non-vanishing term of the expansion of $g(t)$. This generalizes the result in §2.

The complete set of symmetries listed in §3 is obtained only if the mapping

* A mapping function $\phi(t)$ may be regarded as derived from a generating function $\psi(t)$ defined for $0\leqq t<1$ and related graphically to $\phi(t)$ as follows: $\phi(t)$ is the abscissa of the point in which the line through the points $(0, 1)$ and $(t, 0)$ in the ordinary plane cuts the graph of $y=\psi(x)$. Taking $\psi(t)=t$ gives the function $\phi(t)=t/(1+t)$ used in our examples.

function satisfies the identity (6). If this identity holds,† the graph of $xy=1$ is still the pair of straight-line segments shown in Figure 2, since $xy=1$ will then transform into $x'+y'=1$ or $x'+y'=-1$ according as x and y are both positive or negative. Furthermore, if the identity holds, it may be proved* that if $y=f(x)$ is any differentiable function such that $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to\infty} f'(x)=m>0$, then the drawn slope of $y=f(x)$ at $(+\infty, +\infty)$ is $1/m$.

8. Condensed coordinate systems. In the plane or in space, with any of the usual coordinate systems, the distance coordinate scales may be condensed. The hyperbolic paraboloid $z=x^2-y^2$ shows up nicely when drawn in rectangular condensed space. A polar condensed plane is obtained by condensing the distance coordinate while leaving the angular coordinate unchanged. A cylindrical condensed space or a spherical condensed space may also be obtained in this way. The polar condensation and its spatial extensions seem generally unsatisfactory as analytical aids, but may occasionally be useful when the graphs to be displayed have marked polar symmetry. For special kinds of geometrical or analytical illustrations, one distance coordinate may be shrunk while leaving another unchanged.

† Paul Erdös pointed out to me that a $\phi(t)$ defined only on $0\leqq t\leqq 1$ could be selected for which $\phi(0)=0$, $\phi(1)=1/2$, $\phi'(t)>0$, $t\phi'(t)$ increasing, and that $\phi(t)$ could then be defined for $1<t<+\infty$ by (6), the result being a mapping function satisfying the identity as well as the earlier stated minimum requirements.

* The proof makes a nice exercise in the application of l'Hospital's Rule, suitable for an advanced student.

II. A CIRCULAR MODEL OF THE EUCLIDEAN PLANE

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1. The transformation T. Consider the lower half, *i.e.*, the half for which $z<1$, of the unit-sphere

$$x^2+y^2+z^2-2z=1, \tag{1}$$

with center at $A(0, 0, 1)$ and tangent to the xy-plane at the origin. With A as center, project the xy-plane onto this hemisphere so that an arbitrary point P of the plane goes into the point Q of the hemisphere. In this way the entire xy-plane is mapped topologically onto the entire hemisphere exclusive of its bounding great circle. Now project this hemisphere orthogonally onto the xy-plane, so that the arbitrary point Q of the hemisphere goes into the point P' of the plane. The aggregate of all the points P' is the interior of the unit-circle $x^2+y^2=1$ in the xy-plane, which circle we denote by γ. The resultant of the two projections is a topological mapping of the entire xy-plane onto the interior of γ such that an arbitrary point P of the plane goes into the point P' interior to γ.

Denote this mapping by T.

In the projection with A as center each straight line of the plane goes into a semi-great-circle on the hemisphere, exclusive of its endpoints, and this semi-great-circle, in turn, projects orthogonally into a semi-ellipse, exclusive of its endpoints, whose major axis is a diameter of γ. Let E denote the aggregate of all such semi-ellipses. The mapping T thus sends the set of all straight lines of the plane into the set E. For this statement to be entirely correct the term "semi-ellipse" must be extended, as we hereby do, to include any diameter of γ, for T sends each straight line $y=mx$ into such a diameter.

It is easy to see that under T each straight line through the origin O goes into the diameter of γ contained in that line, and each circle with center O goes into a smaller circle with center O and interior to γ. Thus the effect of T is to compress the plane radially into the interior of γ.

By using equation (1), together with the fact that the parametric equations of the straight line AP are

$$x = at, \qquad y = bt, \qquad z = -t + 1, \tag{2}$$

we readily obtain

$$x' = \frac{x}{\sqrt{x^2 + y^2 + 1}}, \qquad y' = \frac{y}{\sqrt{x^2 + y^2 + 1}} \tag{3}$$

as the equations of the mapping T, and hence

$$x = \frac{x'}{\sqrt{1 - x'^2 - y'^2}}, \qquad y = \frac{y'}{\sqrt{1 - x'^2 - y'^2}} \tag{4}$$

as the equations of the inverse mapping T^{-1}. All the previously stated properties of T can now be easily verified analytically.

2. The semi-ellipses E. We now wish to consider the problem of locating the semi-ellipse E into which T sends a given straight line l. To locate E exactly we could, of course, assume an equation for l, transform it into the equation of E, and study the latter equation. This procedure, though direct, lacks the geometric appeal and insight of the method we now prefer to follow.

The central projection from A sends l into a semicircle m on the hemisphere, the diameter d of this semicircle going through A. Because of this projection l and d are parallel, being in the same plane through A yet not meeting since they are in the planes $z=0$ and $z=1$, respectively. The semicircle m and its diameter d project orthogonally into the semicircle E and its major axis d', so that d and d' are parallel. Hence l and d' are parallel. That is, the straight line l is parallel to the major axis of the corresponding semi-ellipse E.

Now let B be the foot of the perpendicular from O to l, and B' the point into which T sends B. Then O, B, B' are collinear since T is a radial transformation. Since B is the point of l nearest O it follows from next to the last paragraph of

section 1 that B' is the point of E nearest O, in which case B' must be the extremity of the minor axis of E, and $|\overline{OB'}|$ the length of this axis.* To determine the length of this axis, let p and p' denote the distances $|\overline{OB}|$ and $|\overline{OB'}|$, respectively, let (x, y) and (x', y') be the coordinates of B and B', respectively, and let $ax+by+c=0$ be the equation of l. Since p is the length of the normal from O to l we have

$$p = \frac{|c|}{\sqrt{a^2+b^2}},$$

so that

$$\frac{p^2}{p^2+1} = \frac{c^2}{a^2+b^2+c^2}. \tag{5}$$

Also

$$\begin{cases} p^2 = \overline{OB}^2 = x^2 + y^2 \\ p'^2 = \overline{OB'}^2 = x'^2 + y'^2, \end{cases} \tag{6}$$

while the equations of T^{-1} permit us to write

$$x'^2 + y'^2 = \frac{x^2+y^2}{x^2+y^2+1}. \tag{7}$$

Substitution of (6) in (7) gives

$$p'^2 = \frac{p^2}{p^2+1},$$

and substitution of (5) in this gives

$$p'^2 = \frac{c^2}{a^2+b^2+c^2}.$$

Hence the length of the minor axis of E is given by

$$p' = \frac{|c|}{\sqrt{a^2+b^2+c^2}}.$$

We still need to determine the quadrant in which the minor axis lies. Since B' is between O and B, this axis and segment OB lie in the same quadrant, which is readily determined from the normal form of l, or merely from inspection of the equation $ax+by+c=0$. Thus, if c is always taken negative, the following table gives the proper quadrant:

* By "minor axis of E" is meant the semi-minor axis of the complete ellipse of which E is a part.

a	b	quadrant
+	+	1
−	+	2
−	−	3
+	−	4

Using the preceding results we can draw the major and minor axes and thus provide ourselves with a good idea of the shape and location of E. Of course, knowing these axes, as many points of E as may be desired can be constructed by the familiar straightedge and compass method which utilizes two concentric circles. Should one wish to sketch E only by using the major axis, minor axis, and latera recta, it may be noted that the distance from O and the length of each latus rectum† are, respectively,

$$\sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}} \quad \text{and} \quad \frac{c^2}{a^2+b^2+c^2}.$$

3. A topological model of the Euclidean plane. Since T is a topological mapping, the points and semi-ellipses E within γ have the same topological properties as do the points and straight lines of Euclidean plane geometry. Thus any two points within γ determine a unique semi-ellipse E, any two semi-ellipses E meet in one point or not at all, a semi-ellipse E is a continuous curve that does not intersect itself, and so forth. The interior of γ viewed in this way is therefore a topological model of the Euclidean plane.

The general value of this model lies in the fact that, being bounded, it puts the entire Euclidean plane before our eyes in miniature, as it were, giving us a topological view of the plane in the large that is impossible with the usual unbounded Cartesian model. A straight line, *i.e.*, a semi-ellipse E, can be visualized in its entirety, and the same is true of any other unbounded curve. A family of parallel straight lines appears to us completely as a family of semi-ellipses E no two of which meet since they all have the same major axis, *i.e.*, the same diameter of γ. Clearly this model can serve as a handy means of exhibiting in miniature the intersection properties of straight lines, as well as their other topological properties, and it offers the possibility that these things may also be done to advantage with non-linear unbounded curves. It is worth noting that this model is somewhat analogous to a certain topological model of the hyperbolic plane in which the points interior to a circle and the chords of this circle are defined as hyperbolic points and straight lines, respectively.*

Over and above this general value our model possesses the special advantage

† By "latus rectum" of E is meant half of the latus rectum of the complete ellipse of which E is a part.

* See, for example, Richard Baldus, *Nichteuklidische Geometrie*, chapter IV.

that it can be enlarged in a simple and concrete manner so as to produce a topological model of the plane of elliptic or projective geometry. This is done merely by adding to our model the points on the circle γ, with the understanding that each pair of antipodal points is to be regarded as a single point, and that each semi-ellipse E thus acquires a single new point. This vivid and natural way of introducing ideal points contrasts strongly with the usual method employing the unbounded Cartesian model. In this latter method, as usually presented in books on modern geometry, we are expected to believe that the ideal point on a straight line is a definite point that is reached by going infinitely far either to the left or to the right along the unbounded line! We are furthermore expected to believe that there is exactly one ideal point corresponding to each slope in the plane, and that all these ideal points constitute an ideal straight line. The unbounded Cartesian model is certainly worthless as a pictorial basis for such beliefs.

4. An isometric model of the Euclidean plane. By a very simple device our topological model can be made to possess all the metric properties of the Euclidean plane. First, instead of thinking of the points interior to the circle γ as possessing coordinates (x', y'), where $x'^2+y'^2<1$, we can think of them as having coordinates (x, y), where x and y are any real numbers, merely by regarding the equations for T and T^{-1} as equations for changing coordinates. Thus each point (x', y') interior to γ acquires new coordinates (x, y) in accordance with these equations, so that, for example, the point $x'=0$, $y'=1/2$ acquires the new coordinates $x=0$, $y=1/\sqrt{3}$. Second, having thus changed coordinates, we continue to use the same distance formula $ds^2=dx^2+dy^2$, so that, for example, the distance between the points with new coordinates (2, 0) and (5, 4) is 5.

It follows that our model possesses all the metric properties of Euclidean plane geometry, except, of course, that we cannot continue to visualize things in the conventional manner. We must visualize our straight lines as semi-ellipses E, although they still have equations of the form $ax+by+c=0$. To draw the locus of such an equation we simply draw a certain semi-ellipse E, as explained earlier. The angle between two straight lines must be visualized as the angle between two semi-ellipses E, but the computation of this angle from the equations of these semi-ellipses is exactly the same as in ordinary Cartesian geometry.

An alternative method of converting our topological model of the Euclidean plane into an isometric model would be to let the points interior to γ retain their usual (x', y') coordinates, where $x'^2+y'^2<1$, but to adopt a new formula for distance. The new formula, however, turns out to be so complicated as to make this alternative approach quite impractical.

III. THE USE OF CONDENSED GRAPHS IN ANALYTIC GEOMETRY

K. O. MAY, Carleton College

By constructing appropriate graph paper we may use the circular model of the Euclidean plane described in the previous paper to deal with unbounded curves at the elementary level. For cartesian graphs the coordinate curves $x=a$ are semi-ellipses with major axes coincident with the diameter of the circle and semi-minor axes given by $a[1+a^2]^{-1/2}$, and similarly for the curves $y=b$. In Fig. 1 the coordinate ellipses are shown for x, $y=\pm 10$, ± 20. For polar graphs the coordinate curves $r=a$ are circles whose radii are given by the previous formula, and the curves $\theta=b$ are diameters whose slopes are given by $\tan b$. Circles for $r=10$, 20, 30, and 40 are shown in Fig. 2.

Graph paper may be prepared easily by drawing in one quadrant one set of coordinate curves. This may then be used to trace both curves in all quadrants on a spirit duplicator master unit. The result will be serviceable even if rather inelegant. The writer will be glad to send a sample to anyone interested.

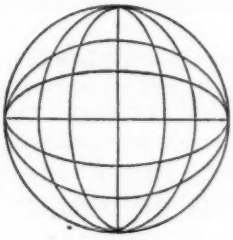

Fig. 1

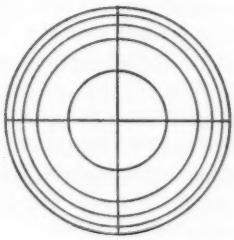

Fig. 2

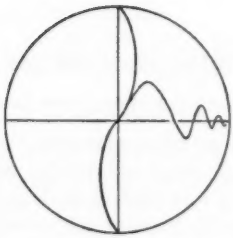

Fig. 3

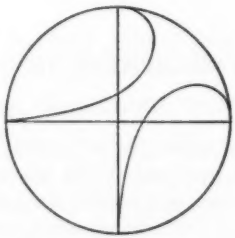

Fig. 4

Once the graph paper is constructed it may be used without reference to its origin in a transformation of the Euclidean plane. Indeed it is a perfectly legitimate independent model of the set of all ordered pairs of real numbers. Of course the numbers assigned to the coordinate ellipses may be changed by any

multiplicative factor. Since there is practically no distortion from the familiar cartesian coordinates in the central region (note the central "square" in Fig. 1), we may choose the scale so as to show as much as we like of a curve in the familiar shape. On the other hand we may choose the scale so as to compress much of the curve near the origin and show greater detail farther out. The graphs of $y=\tan x$ and $y=\sin x$ in Fig. 3 and of $y=e^x$ and $y=\log x$ in Fig. 4 exhibit the familiar forms near the origin.

The following features of this graph paper are easily verified.

1. Symmetries about the origin and about lines through it are undisturbed. In particular, as illustrated in Fig. 4, the relation between the graphs of a function and its inverse remains.

2. The elliptical graph of a linear equation can easily be drawn by rotating the graph paper until one of the coordinate ellipses coincides with it.

3. Slopes may be estimated and bend points located with reference to neighboring coordinate curves.

4. Periodic curves appear to recede in perspective as in Fig. 3.

5. Curves asymptotic to straight lines approach the boundary along ellipses as illustrated for $y=\tan x$ in Fig. 3.

6. Curves without asymptotes approach the boundary in a characteristically different way. Thus curves such as $y=e^x$ (Fig. 4) and $y=x^2$ cross every coordinate ellipse and so appear to come in "along the circumference." These properties may be emphasized by change of scale, but we are not suggesting a graphical method to replace analytic studies of curves "at infinity." However, this and the previous property make the graph paper intuitively very satisfactory for dealing with unbounded curves.

CONCERNING THE RECIPROCAL OF A PRIME

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1. On evaluating the reciprocal of a prime number, one often comes across a figure in the quotient which gives a corresponding remainder equal to that figure in the quotient. Again for primes like $p=19$ and 29 such pairs of figures in the quotient and corresponding remainder which are equal are exactly 9. It is therefore interesting to study the conditions under which such properties can be noticed.

2. Let $q(a)$ and $r(a)$ be the quotient and its corresponding remainder in the ath place in the period of the recurring decimal for $1/p$. Then, we have:

$$(2.1) \qquad \frac{10r(a)}{p} = q(a+1) + \frac{r(a+1)}{p}, \qquad 0 \leqq q(a) \leqq q;\ 0 < r(a) < p.$$

For any $q(a)=r(a)$ we must have $p\cdot q(a)+r(a)=10r(a-1)$, *i.e.*,

$$r(a)\cdot(p+1)=10r(a-1). \tag{2.2}$$

From (2.2) it is possible to deduce the following results.

3. (a) *For primes of the form $p=10n+s$ where $s=1, 3, 7$, $q(a)=r(a)$ only if $r(a)=5$ has a solution.*

(b) *For primes of the form $p=10n+9$, $q(a)=r(a)=b$ is possible for all values of b, $(1\leqq b\leqq 9)$ except for those values of b for which $r(a-1)=b(n+1)$ does not have integral solutions.*

(c) COROLLARY: *For primes of the form $p=10n+9$, $q(a)=r(a)=b(1\leqq b\leqq 9)$ where the period of the recurring decimal has $p-1$ figures.*

Indications of proofs.

(a) In (2.2) putting $p=10n+s$ we get

$$r(a)\cdot(10n+s+1)=10\cdot r(a-1). \tag{3.1}$$

i.e.,

$$r(a)\cdot(5n+d)=5\cdot r(a-1) \qquad \text{where } 2d=s+1;\ d=1, 2, 4. \tag{3.2}$$

By taking residues (mod 5) one easily sees that (3.2) has only one solution *viz.*,

$$r(a)=5, \qquad 9\geqq q(a)=r(a)>0, \tag{3.3}$$

under the condition that $q(a)=5$ has a solution. When the period $=p-1$, $r(a)$ runs through all values such that $0<r(a)\leqq p-1$.

(b) In (2.2) putting $p=10n+9$ we get

$$r(a)\cdot(n+1)=r(a-1). \tag{3.4}$$

Therefore, for all values of $r(a-1)$ which are divisible by $(n+1)$, equation (3.4) has integral solutions.

When the period $=p-1$ it is easily seen that such pairs of solutions of (3.4) are exactly equal to 9, and $q(a)=r(a)=b$ where b assumes all values from 1 to 9.

Examples.

Below are given two examples to illustrate the properties mentioned above.

(i) $p=17$; this is of the form $p=10\cdot 1+7$ and period $=16$. From §3 (a) there exists only one pair $q(a)=r(a)=5$. This is shown in the table given below.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$q(a)$	5	8	8	2	3	5	2	9	4	1	1	7	6	4	7	0
$r(a)$	15	14	4	6	9	5	16	7	2	3	13	11	8	12	1	10

(ii) $p=19$; this is of the form $p=10\cdot 1+9$ and period $=18$. From §3(a) there

exist exactly 9 pairs $q(a)=r(a)=b$ where $1 \leqq b \leqq 9$. This is illustrated in the table given below.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$q(a)$	5	2	6	3	1	5	7	8	9	4	7	3	6	8	4	2	1	0
$r(a)$	5	12	6	3	11	15	17	18	9	14	7	13	16	8	4	2	1	10

MATHEMATICAL TEACHING IN UNIVERSITIES

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The following is the outline of a lecture once given by the author at a joint meeting of the Nancago Mathematical Society and of the Poldavian Mathematical Association. It is printed here at the editor's request, as the principles stated there seem to be of general application.

1. Improvements in the mathematical teaching in Poldavian Universities depend largely upon general improvements in the educational system in Poldavia. Mathematicians should devote themselves to the task of making such improvements as lie within their power at present, and thus contributing their share towards general reforms, which in turn will enable them to make further progress.

2. No satisfactory results can be achieved unless reforms are made both in school-teaching and in University teaching. So far as school-teaching is concerned, the efforts of mathematicians in the country should be mainly directed towards necessary changes in the curricula and towards the training of better teachers.

3. University teaching in mathematics should: (a) answer the requirements of all those who need mathematics for practical purposes; (b) train specialists in the subject; (c) give to all students that intellectual and moral training which any University, worthy of the name, has the duty to impart.

These objects are not contradictory but complementary to each other. Thus, a training for practical purposes can be made to play the same part in mathematics as experiments play in physics or chemistry. Thus again, personal and independent thinking cannot be encouraged without at the same time fostering the spirit of research.

4. The study of mathematics, as well as of any other science, consists in the acquisition of useful reflexes and in that of independent habits of thought. The acquisition of useful reflexes should never be separated from the perception of their usefulness.

It follows that problem-solving should never be practised for its own sake; and particularly tricky problems must be excluded altogether. The purpose of problems is twofold; either to drill the student in the application of some method of special importance, or to develop his originality by guiding him along some new path. Drill is essentially a school-method, and ought to become unnecessary at the final stages of University teaching.

5. Rigor is to the mathematician what morality is to man. It does not consist in proving everything, but in maintaining a sharp distinction between what is assumed and what is proved, and in endeavoring to assume as little as possible at every stage.

The student should therefore be gradually accustomed, by means of startling examples, to question the truth of every unproved proposition, until at last he is able to deduce from the ordinary axioms everything that he has learned.

6. Knowledge of a proof means the understanding of its machinery and the ability to reconstruct it. This implies: (a) perfect correctness in the definitions; (b) a faculty of connecting a given question with the general ideas underlying it; (c) a perception of the logical nature of any proof.

The teacher should therefore always follow, not the quickest nor even the most elegant method, but the method which is related to the most general principles. He should also point out everywhere the relation between the various elements of the hypothesis and the conclusion; students must be accustomed to draw a sharp distinction between premises and conclusion, between necessary and sufficient conditions, between a theorem and its converse.

7. The teaching of mathematics must be a source of intellectual excitement. This can be achieved, at the higher stages, by taking the student to the brink of the unknown; at earlier stages, by making him solve for himself questions of theoretical or practical importance.

This is the method followed in the "seminars" of the German Universities, first organized by Jacobi a century ago, and even now the most prominent feature of the German system; division of labor between students in the study of a given group of questions is a common practice in these seminars, and proves to be a powerful incentive to work.

8. Theoretical lectures should neither be a reproduction of nor a comment upon any text-book, however satisfactory. The student's notebook should be his principal text-book.

In fact, taking down notes intelligently (not under dictation) and working them out carefully at home should be considered as an essential part of the student's work; and experience shows that it is not the least useful part of it.

9. The right of any topic to form part of any curriculum is to be tested according to: (a) its importance for modern mathematics or for the applications of mathematics to modern science or technique; (b) its relations with other branches of the curriculum; (c) the intrinsic difficulty of the ideas underlying it.

This involves a revision of the present curriculum. For instance, the idea of

function, the process of differentiation and integration, should appear at an early stage, because of their enormous importance both for the theory and for the most ordinary practice. Because of its practical importance, numerical calculation, and all the devices connected with it, would seem to deserve a far more prominent place in elementary teaching than they receive at present.

MATHEMATICAL NOTES

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AN EXPRESSION FOR THE EULER ϕ-FUNCTION

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Let a and x be non-negative integers, $x>0$. If we divide a by x we have:

$$a = x \cdot Q + r$$

where $Q=[a/x]$ is the integral quotient and r is the remainder in the division. Then:

$$r = a - x \cdot [a/x].$$

(The symbol [ ] will represent the bracket function throughout this article and not a parenthesis. By definition:

$[u]$ = the greatest integer not greater than u.)

For r we have the inequalities:

$$0 \leqq r = a - x \cdot [a/x] < x.$$

1. We construct the function:

$$G(a, x) = \left[\frac{1}{a - x \cdot [a/x] + 1}\right]. \tag{1}$$

Obviously $G(a, x)=1$ if, and only if, x divides a exactly; otherwise $G(a, x)=0$; a, x are non-negative integers, $x>0$.

2. Next we construct the function:

$$G(a, b, x) = \left[\frac{1}{a - a \cdot [a/x] + b - b \cdot [b/x] + 1}\right]; \tag{2}$$

a, b, x are non-negative integers, $x>0$. Comparing (2) with (1) it is easily seen that $G(a, b, x)=1$ if, and only if, x divides exactly both a and b. Otherwise

$G(a, b, x) = 0$.

3. Continuing, we consider now the expression:

$$L(a, b) = \sum_{x=2}^{a+b} G(a, b, x).$$

Examining the individual terms of the sum on the right we see that each term has the value 1 if x is a common divisor of a and b and otherwise the term has the value 0. Clearly, then, $L(a, b)$ represents the number of common divisors of a and b exclusive of unity. Notice the elasticity of the upper limit over the summation sign which can be taken as any integer not less than the smaller of the two numbers a and b.

We denote the greatest common divisor of a and b, as usual, by (a, b). Notice now that if $(a, b) = 1$, *i.e.*, if a and b are relatively prime they have no common divisor greater than unity and hence $L(a, b) = 0$. If, however, $(a, b) > 1$ and a and b are not relatively prime then clearly, $L(a, b) > 0$.

4. It follows then that the function:

$$L'(a, b) = \left[\frac{1}{1 + L(a, b)}\right] = \left[\frac{1}{(a, b)}\right]$$

is zero when $(a, b) > 1$, *i.e.*, when $L > 0$; while $L'(a, b) = 1$ when, and only when, $(a, b) = 1$, *i.e.*, when $L = 0$.

5. Finally we arrive at the result:

$$\phi(a) = \sum_{b=1}^{a-1} L'(a, b)$$

where a is a positive integer > 1. The truthfulness of this becomes at once obvious when we examine the terms of the sum on the right. A term of the sum is zero when a and b have a common divisor. If there is no common divisor the term has the value one and in this case a and b are relatively prime. Thus the sum counts up for us all integers less than a and relatively prime to it thus yielding Euler's totient.

THE NUMBER OF MULTINOMIAL COEFFICIENTS

PAUL ERDÖS, National Bureau of Standards, and IVAN NIVEN, University of Oregon

The problem is to find the number of multinomial coefficients

$$\frac{n!}{i_1! i_2! \cdots i_r! (n - k)!}, \qquad \sum_{j=1}^{r} i_j = k, \tag{1}$$

which are less than x, excluding the cases $r = 1 = i_1$ and $r = 1 = k$ for which (1) assumes the value n. The values of (1) are thus restricted by

$$i_1 \leqq i_2 \leqq \cdots \leqq i_r \leqq n-k \leqq n-2. \tag{2}$$

One of the writers [1] proved earlier that these multinomial coefficients have density zero; we now prove the following stronger result.

THEOREM. *The number of multinomial coefficients* (1) *which satisfy* (2) *and which are less than a fixed* $x>0$ *is* $(1+\sqrt{2})x^{1/2}+o(x^{1/2})$.

To prove this we divide the values (1) into 3 classes and treat each class separately. The first class, $f_1(x)$ in number, contains those having $k=2$; the second class, $f_2(x)$ in number, those with $3\leqq k\leqq n/2$; the third class, $f_3(x)$ in number, those with $k>n/2$. We prove that

$$f_1(x) = (1+\sqrt{2})x^{1/2} + o(x^{1/2}), \qquad f_2(x) = o(x^{1/2}), \qquad f_3(x) = o(x^{1/2}), \tag{3}$$

which will establish the theorem.

Class 1. For $k=2$ the values (1) are the two types, $n(n-1)/2$ and $n(n-1)$. Now $n(n-1)<x$ for $x^{1/2}+o(x^{1/2})$ values of n, and $n(n-1)/2<x$ for $(2x)^{1/2}+o(x^{1/2})$ values of n. We must eliminate duplicates, that is cases where

$$n(n-1)=m(m-1)/2. \tag{4}$$

We show that (4) has at most $c \log x$ solutions $<x$, and this will establish the first equation (3). Solving (4) for m we find that solutions exist if and only if $8n^2-8n+1$ is a perfect square, say u^2, and replacing $2n-1$ by z, we have (4) reduced to $u^2-2z^2=1$. The positive integral solutions of this equation are given (*cf.* [3]) by $u+z\sqrt{2}=(3+2\sqrt{2})^r$ for $r=1, 2, \cdots$, and the number of these less than x is of the order of $c \log x$.

Class 2. For any fixed k and n, the equation $k=\sum i_j$ indicates that the number of values of (1) is $p(k)$, the number of partitions of k into positive integers. The smallest of these $p(k)$ values is $\binom{n}{k}$, and so the admissible values of n and k will satisfy $\binom{n}{k}<x$. Now

$$\binom{n}{k} = \prod_{j=0}^{k-1} \frac{n-j}{k-j} > \left(\frac{n}{k}\right)^k. \tag{5}$$

Hence the admissible values of n and k satisfy $\left(\frac{n}{k}\right)^k<x$ or $n<kx^{1/k}$. Thus for each k the maximum number of values of n is $kx^{1/k}$ and so

$$f_2(x) < \sum_k (kx^{1/k})p(k), \tag{6}$$

the sum ranging over the admissible values of k. By definition of $f_2(x)$ the smallest k is $k=3$. To get an upper bound of k in terms of x we observe that the largest k corresponds to $n=2k$. Using $\binom{n}{k}<x$ again, we have that admissible values of k satisfy $\binom{2k}{k}<x$ and so satisfy $2^k<x$ since $2^k<\binom{2k}{k}$ by (5). Thus the range in the sum (6) can be taken as $k=3$ to $k=c \log x$.

Now [2] $p(k)<e^{c_1\sqrt{k}}$, so $p(c \log x)<e^{c_1(c \log x)^{1/2}}<x^\epsilon$ for arbitrary $\epsilon>0$ with x

sufficiently large. Maximizing each part of (6) gives

$$f_2(x) < (c \log x)(c \log x)x^{1/2}x^{\epsilon} = o(x^{1/2}).$$

Class 3. Every value (1) in this class is clearly

$$\geqq \frac{n!}{\left[\frac{n}{2}\right]!\left\{n - \left[\frac{n}{2}\right]\right\}!}.$$

Thus each admissible value of n satisfies $n \leqq 2h+1$ where h is chosen so that $\binom{2h}{h}$ exceeds x. Replacing $\binom{2h}{h}$ by 2^h as previously we see that $n < c \log x$. For any fixed n the number of values of (1) is maximized by $p(n)$. Thus $f_3(x) < c \log x \cdot p(c \log x) = o(x^{1/2})$, and the proof of (3) is complete.

A more careful analysis would improve the theorem to yield the estimate

$$(1 + \sqrt{2})x^{1/2} + c_3x^{1/3} + \cdots + c_mx^{1/m} + o(x^{1/m})$$

for every m. This could be proved by isolation of the cases $k = 2, 3, \cdots, m$ for special treatment, where here we stopped at $k = 2$.

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1. Ivan Niven, The asymptotic density of sequences, Bull. Amer. Math. Soc., vol. 57, 1951, pp. 420–434, Theorem 2.

2. G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2), vol. 17, 1918, pp. 75–115; or, Paul Erdös, On an elementary proof of some asymptotic formulas in the theory of partitions, Annals of Math. (2), vol. 43, 1942, pp. 437–450.

3. Nagell, T., Introduction to Number Theory, John Wiley (1952), Theorem 104, p. 197.

EINSTEIN NUMBERS

G. A. Baker, Jr., California Institute of Technology

H. T. Davis [1] suggests that we define an operation $\oplus$ as follows:

$$a \oplus b = \frac{a + b}{1 + \frac{ab}{c^2}}.$$

Then c has two of the three properties ($A + A = A$; $A + n = A$, n any finite number; and $A^p = A$, p any integer) ascribed to infinity by Cantor, namely

$$c \oplus c = \frac{c + c}{1 + \frac{c^2}{c^2}} = \frac{c + c}{2} = c$$

and

$$c \oplus n = \frac{c+n}{1+\frac{cn}{c^2}} = c\left(\frac{n+c}{n+c}\right) = c.$$

Einstein makes use of this operation in his theory of relativity. He identifies a and b with relative velocities. It will be interesting to investigate further properties of his numbers.

If R is the set of all real numbers and S is the set of all Einstein numbers (real numbers whose absolute value is less than c), then the function $c \tanh x$ has the following properties:

(1) the domain of $c \tanh x$ is R
(2) the range of $c \tanh x$ is S
(3) $c \tanh x$ is bi-unique
(4) $c \tanh (x+y) = c \tanh x \oplus c \tanh y$, where x and y belong to R, for

$$c \tanh (x+y) = \frac{c \tanh x + c \tanh y}{1+\frac{(c \tanh x)(c \tanh y)}{c^2}} = c \tanh x \oplus c \tanh y.$$

S is closed with respect to $\oplus$. If a and b belong to S then by (2) and (4) we can write $a = c \tanh x$, $b = c \tanh y$, $a \oplus b = c \tanh x \oplus c \tanh y = c \tanh (x+y)$. Therefore, since $|c \tanh (x+y)| < c$, it follows that $|a \oplus b| < c$.

Hence the abelian group consisting of the set, R, of all real numbers and the operation, ordinary addition, is isomorphic under the transformation $c \tanh x$ to the algebraic system consisting of the set, S, and the operation $\oplus$. Therefore S and $\oplus$ form an abelian group.

This isomorphism leads us to define a new operation $\odot$ as follows:

$$a \odot b = c \tanh\left[\left(\tanh^{-1}\frac{a}{c}\right)\left(\tanh^{-1}\frac{b}{c}\right)\right].$$

The abelian group consisting of the set, R_1, of all real numbers except zero and the operation, ordinary multiplication, is isomorphic under the transformation $c \tanh x$ to the algebraic system consisting of the set, S_1, of all Einstein numbers except zero and the operation $\odot$. Therefore, S_1 and $\odot$ form an abelian group. For:

(1) the domain of $c \tanh x$ is R
(2) the range of $c \tanh x$ is S
(3) $c \tanh x$ is bi-unique
(4) $c \tanh (xy) = c \tanh x \odot c \tanh y$
since

$$c \tanh x \odot c \tanh y = c \tanh \left[\left(\tanh^{-1} \frac{c \tanh x}{c}\right)\left(\tanh^{-1} \frac{c \tanh y}{c}\right)\right]$$

$$= c \tanh xy$$

(5) S_1 is closed with respect to $\odot$ since

$$\left| c \tanh \left[\left(\tanh^{-1} \frac{a}{c}\right)\left(\tanh^{-1} \frac{b}{c}\right)\right]\right|$$

is less than c and is not equal to zero. It equals zero only if $\tanh^{-1} a/c$ or $\tanh^{-1} b/c$ equals zero. However, $\tanh^{-1} x$ equals zero only if x equals zero. Therefore, since zero does not belong to S_1,

$$c \tanh \left[\left(\tanh^{-1} \frac{a}{c}\right)\left(\tanh^{-1} \frac{b}{c}\right)\right]$$

does not equal zero.

The operation $\odot$ is distributive with respect to $\oplus$ for if a, b, d belong to S we may write $a = c \tanh u$, $b = c \tanh v$, $d = c \tanh w$. Then,

$$\begin{aligned} a \odot (b \oplus d) &= c \tanh u \odot (c \tanh v \oplus c \tanh w) \\ &= c \tanh u \odot c \tanh (v + w) \\ &= c \tanh (u(v + w)) \\ &= c \tanh (uv + uw) \\ &= c \tanh (uv) \oplus c \tanh (uw) \\ &= (c \tanh u \odot c \tanh v) \oplus (c \tanh u \odot c \tanh w) \\ &= (a \odot b) \oplus (a \odot d). \end{aligned}$$

We note that c also has the third property of infinity, namely $c^p = c$. For

$$c \odot c \odot \cdots \odot c \odot c = (c \odot c \odot \cdots \odot c) \odot (c \odot c)$$

and

$$c \odot c = c \tanh \left[\left(\tanh^{-1} \frac{c}{c}\right)\left(\tanh^{-1} \frac{c}{c}\right)\right].$$

Since $\tanh^{-1} 1 = \infty$, $(\tanh^{-1} 1)^2 = \tanh^{-1} 1$, then $c \odot c = c \tanh \tanh^{-1} 1 = c$. Hence, by induction, c times itself any integral number of times equals c.

The Einstein numbers form a field under the operations $\oplus$ and $\odot$. The number, c, which may be identified with the velocity of light, acts for this field as infinity does for the real field. The real field is isomorphic to this field under the transformation, $c \tanh x$.

Reference

1. H. T. Davis, College Algebra, Prentice-Hall, Inc., N.Y., 1940, p. 351.

CLASSROOM NOTES

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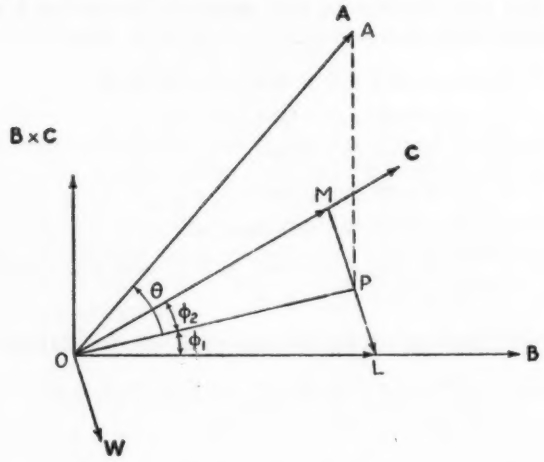
VECTOR TRIPLE PRODUCT

C. I. LUBIN, University of Cincinnati

The standard expression for the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ in terms of the vectors **b** and **c** is usually obtained by a quite satisfactory analytical procedure. (See: *Vector and Tensor Analysis*, Brand; New York, 1947, p. 40.) However a direct geometrical derivation might be of more appeal to some minds and perhaps also would be more in keeping with the spirit of vector algebra. It is the purpose of this note to present a direct geometrical construction for this product. (For another geometrical treatment, see Coe and Rainich, this MONTHLY, vol. 56, 1949, p. 175.)



First note that the above product can be written as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = abc\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

where a, b, c are the magnitudes of the vectors **a**, **b**, **c** and **A**, **B**, **C** are unit vectors along **a**, **b**, **c** respectively. We are thus concerned with the problem of finding an expression for the vector $\mathbf{w}=\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. Let these vectors in one of several possible configurations be as shown in the diagram, starting at O, with **B** and **C** in a horizontal plane. The vector product $\mathbf{B}\times\mathbf{C}$ will be a vertical vector of magnitude $\sin\phi$ where $\phi=\angle(\mathbf{B},\mathbf{C})$. Let P be the projection on the plane of **B** and **C** of the end point A of **A**. The triple product **w** will then be in the plane

of $\mathbf{B}$ and $\mathbf{C}$, perpendicular to OP as shown, and of magnitude

$$|\mathbf{w}| = \sin\phi\cos\theta = OP\sin\phi$$

where $\theta = \angle(OP, \mathbf{A})$.

Draw through P a line perpendicular to OP, intersecting the vectors $\mathbf{C}$ and $\mathbf{B}$ at M and L respectively. Then, equating the two expressions following for the area of the triangle OML, we get

$$\tfrac{1}{2}OP\cdot ML = \tfrac{1}{2}OM\cdot OL\sin\phi,$$

whence

$$\tfrac{1}{2}OP\cdot ML = \tfrac{1}{2}\overline{OP}^2\sin\phi\sec\phi_1\sec\phi_2$$

where $\phi_1 = \angle LOP$ and $\phi_2 = \angle POM$.
We thus obtain

$$ML = OP\sin\phi\sec\phi_1\sec\phi_2 = |\mathbf{w}|\sec\phi_1\sec\phi_2$$

and since $\overrightarrow{ML}$ and $\mathbf{w}$ are parallel

$$\mathbf{w} = \cos\phi_1\cos\phi_2\overrightarrow{ML}.$$

Now $\overrightarrow{ML} = \overrightarrow{OL} - \overrightarrow{OM} = OP\sec\phi_1\mathbf{B} - OP\sec\phi_2\mathbf{C}$ so that

$$\mathbf{w} = OP\cos\phi_2\mathbf{B} - OP\cos\phi_1\mathbf{C}.$$

Next we observe, since $\mathbf{A} = \overrightarrow{OP} + \overrightarrow{PA}$, that

$$\mathbf{A}\cdot\mathbf{B} = \overrightarrow{OP}\cdot\mathbf{B} + \overrightarrow{PA}\cdot\mathbf{B} = OP\cos\phi_1,$$

and similarly, that

$$\mathbf{A}\cdot\mathbf{C} = OP\cos\phi_2.$$

We thus obtain the expansion for the product $\mathbf{w}$,

$$\mathbf{w} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$

and finally we get the standard expansion for the vector triple product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = abc\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

It may be remarked here that the establishment of the distributive law for the vector product, say $(\mathbf{a}+\mathbf{b})\times\mathbf{u}$ can be made to depend on the distributive law for the scalar product by replacing $\mathbf{u}$ by the vector product of two appropriate vectors and then using the above standard expansion for the vector triple product.

ON THE REMAINDER IN FORMULAS OF NUMERICAL INTERPOLATION, DIFFERENTIATION AND QUADRATURE

HARRY GOHEEN, Iowa State College

W. E. Milne* has developed an approach to the determination of errors in numerical quadrature formulas, differentiation formulas and interpolation formulas based on linear operators. It is the purpose of this note to demonstrate a more elementary approach to this problem which yields the same result and may be given to less mature students.

Following Milne, a formula is called an nth degree formula if it has zero remainder for polynomials of the ith degree for $i=0, 1, 2, \cdots, n$, but not for polynomials of higher degree. The form of such formulas is one of the three:

$$\int_a^b f(x)dx = L_1[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_1, \tag{1}$$

$$f^{(i)}(b) = L_2[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_2, \tag{2}$$

$$f(b) = L_3[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_3, \tag{3}$$

in which L_1, L_2, and L_3 are linear functions of the indicated arguments and r_1, r_2, and r_3 are the remainders. It will be assumed that these formulas are nth degree formulas and that $f(x)$ has an integrable derivative of order $n+1$.

The Taylor expansion, with remainder, of $f(x)$ around $x=a$, is

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt. \tag{4}$$

Then since the formulas (1), (2), and (3) are linear and exact for polynomials of degree n, there result on substitution of (4) into (1), (2), and (3) respectively:

$$r_1 = \int_a^b \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dtdx - L_1\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right], \tag{5}$$

* Milne, W. E. *Numerical Calculus*, Princeton, 1949, pp. 108–116.

$$r_2 = \frac{d^i}{dx^i}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=b}$$

$$(6) \qquad - L_2\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right],$$

$$r_3 = \int_0^{b-a} \frac{(b-a-t)^n}{n!} f^{(n+1)}(a+t)dt$$

$$(7) \qquad - L_3\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right].$$

Thus in order to evaluate the remainders, the only equipment necessary is the equipment to evaluate the expressions

$$(8) \qquad \int_0^{x_j-a} \frac{(x_j-a-t)^n}{n!} f^{(n+1)}(a+t)dt,$$

$$(9) \qquad \int_a^b \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dtdx,$$

and

$$(10) \qquad \frac{d^i}{dx^i}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_j}.$$

The expression (9) is a double integral over the triangle in the (x, t) plane bounded by $x=a$, $x=b$, $t=0$ and $t=x-a$. Inversion of the order of integration yields for (9) the expression,

$$(11) \qquad \int_0^{b-a} \int_{t+a}^b \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dxdt,$$

or

$$(12) \qquad \int_0^{b-a} \frac{(b-a-t)^{n+1}}{(n+1)!} f^{(n+1)}(a+t)dt.$$

In the expression (10) it may be assumed that i is less than n, since a contrary assumption does not yield a practical formula. Then (10) can be reduced by the rules for differentiating a definite integral with respect to a parameter to

$$\int_0^{x_j-a} \frac{(x_j - a - t)^{n-i}}{(n-i)!} f^{(n+1)}(a+t)dt. \tag{13}$$

When (8), (12) and (13) are substituted into (5), (6), and (7) there result formulas for the remainders. In these formulas are integrals of polynomials in t times the common function $f^{(n+1)}(a+t)$. If c is the minimum of the limits and d is the maximum of the limits these formulas may be written as integrals between c and d of a piecewise polynomial function multiplied by $f^{(n+1)}(a+t)$. Denoting the piecewise functions by $g_1(t)$, $g_2(t)$, and $g_3(t)$ there result the formulas of Milne,

$$r_i = \int_c^d g_i(t) f^{(n+1)}(a+t)dt, \qquad i = 1, 2, \text{ and } 3. \tag{14}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1096. *Proposed by L. R. Ford, Jr., University of Illinois*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance, and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

E 1097. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Arcs AB and CD are quadrants of circles tangent externally at their midpoints, E, and such that AC and BD when extended meet perpendicularly in F. A circle is inscribed in the mixtilinear triangle EDB, touching ED in M. G is the projection of M upon EF. Show that triangle MGF is a 3:4:5 right triangle.

E 1098. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

Construct all geometries satisfying the following axioms:
Axiom I. Space S is a set of n points, n a positive integer.
Axiom II. A line is a non-null subset of S.
Axiom III. Any two distinct lines have exactly one common point.
Axiom IV. Every point lies in exactly two distinct lines.

E 1099. *Proposed by W. R. Van Voorhis, Fenn College*

On a certain gambling table there are N squares marked "2 to 1," "3 to 1," $\cdots$, "$N+1$ to 1.". A sum of money is placed on each square and one of the squares is selected as a winner, paying the player at the odds marked on that square. The player loses the amounts placed on the other squares. What is the maximum N for which it is possible for a player to place money so that he can never suffer a net loss?

E 1100. *Proposed by L. E. Ward, Sr., Naval Ordnance Test Station, China Lake, Calif., and L. E. Ward, Jr., University of Nevada*

For $x \geqq 0$ and $t \geqq 2$, prove that $(1+x^t)^{1/t}-(1+x^t)^{-1/t} \leqq x$.

SOLUTIONS

Inscription of a Trapezoid in a Quadrilateral

E 1066 [1953, 331]. *Proposed by A. Zirakzadeh, Oklahoma A. and M. College*

Inscribe a trapezoid in a given quadrilateral such that the bases of the trapezoid will be parallel to one of the diagonals of the quadrilateral and the other two sides will pass, respectively, through two given points.

I. *Solution by W. B. Carver, Cornell University.* Let $PQRS$ be the given quadrilateral, the sides PQ, QR, RS, SP being segments respectively of the lines a, b, c, d. Let $ABCD$ be the trapezoid to be constructed, with vertices A, B, C, D respectively on the lines a, b, c, d, with BC and DA parallel to QS, and with AB and CD passing respectively through the given points H and K.

Take A_1 any point on line a; let the line A_1H cut b at B_1, the line through B_1 parallel to QS cut c at C_1, the line C_1K cut d at D_1, and the line through D_1 parallel to QS cut a at A_1'. If A_1' should happen to coincide with A_1 the problem is solved. In any case, A_1 is projected into A_1' by four steps of projection and section. Now take a second point A_2 on a and project it by similar steps into B_2 on b, C_2 on c, D_2 on d, and A_2' on a. Then on the line a the pairs A_1, A_1' and A_2, A_2' are corresponding pairs of a projectivity, and it is easily seen that this projectivity sends the point Q into itself. The two corresponding pairs and the fixed point Q determine this projectivity and the other fixed point exists and may be constructed by a simple *linear* construction. Let the lines $A_1'B_1$ and $A_2'B_2$ intersect at O; then the line HO cuts the line a in the other fixed point A of the projectivity. We project this point A by the four steps indicated above

into B on b, C on c, D on d, and finally into A on a; and $ABCD$ is then the required inscribed trapezoid.

Three special cases may be noted.

1. The second fixed point of the projectivity may happen to coincide with the other fixed point Q. In this case A and B both coincide with Q and C and D both coincide with S. We hestitate to call $ABCD$ a trapezoid in this case, and prefer to say that there is no solution.

2. It may happen that when the point A has been determined, the line AH will be parallel to b. In this case the line DK will also be parallel to c, and there will be no solution.

3. It may happen that A_1' coincides with A_1 and A_2' coincides with A_2, and then *every* point on the line a is a fixed point of the projectivity. In this case we may take A as any point on the line a and obtain a solution $ABCD$, and hence there are an infinite number of solutions.

In the above discussion the broadest interpretation has been given to the words "quadrilateral," "trapezoid," and "inscribed." The given quadrilateral and the trapezoid need not be convex, the inscribed trapezoid need not lie entirely inside the quadrilateral, the vertices A, B, C, D may lie anywhere on the lines a, b, c, d, and not necessarily on the *segments* PQ, QR, RS, SP. If a more restricted meaning is given to the words, the above method of solution still applies, but very frequently there will be no solution in the more restricted sense.

II. *Solution by R. H. Marquis, Ohio University.* Using the lettering of the previous solution one may easily establish, by elementary geometry, the following construction. Draw SK to cut PR in X. Draw XQ. Draw KY parallel to SQ, to cut XQ in Y. Draw HY to cut PQ in A, QR in B, and PR in Z. Draw KZ to cut RS in C and SP in D. Then $ABCD$ is the required trapezoid.

Also solved by Harry Furstenberg, B. Martin, and the proposer.

Formula Relating Factorials, Permutations, and Combinations

E 1067 [1953, 331]. *Proposed by Tatuya Sasakawa, Kyoto University, Japan*

Show that

$$n! = \sum_{s=0}^{n} \sum_{i=0}^{s} (-1)^{s-i} {}_nC_s \, {}_sP_i.$$

Solution by A. E. Livingston, University of Washington. Let S_n denote the given double sum. Then

$$S_n = n! \sum_{s=0}^{n} \sum_{i=0}^{s} (-1)^{s-i} {}_{n-i}C_{s-i}/(n-i)!$$

$$= n! \sum_{i=0}^{n} \frac{1}{(n-i)!} \sum_{s=i}^{n} (-1)^{s-i} {}_{n-i}C_{s-i}$$

$$= n! \sum_{i=0}^{n} \frac{1}{(n-i)!} \sum_{j=0}^{n-i} (-1)^j {}_{n-i}C_j.$$

Since the last written inner sum represents $(1-1)^{n-i}$ or 1 according as $i \neq n$ or $i=n$, it follows that $S_n = n!$, as asserted in the problem.

Also solved by Harry Furstenberg, H. W. Gould, A. R. Hyde, M. S. Klamkin, David Mandelbaum, William Moser, Michael Skalskyj, Chih-yi Wang, J. V. Whittaker, and the proposer.

Triangles with $s^2 = 2ab$

E 1068 [1953, 331]. *Proposed by W. O. Pennell, Exeter, N. H.*

Given a triangle with sides a, b, c and $s^2 = 2ab$, where s is the semiperimeter. Show that: (1) $s < 2a$, $s < 2b$, (2) $a > c$, $b > c$.

Solution by H. M. Gehman, University of Buffalo. If we substitute $s = (a+b+c)/2$ in $4s^2 = 8ab$, and then subtract $4ab + 4bc$ from each side of the resulting equation, we obtain $(a-b+c)^2 = 4b(a-c)$, from which it follows that $a > c$. Similarly $b > c$.

We shall use now only the weaker hypothesis that $a \geqq b > c$. From the definition of s and from $a < b+c$, we deduce that $s < b+c < 2b$ and that $s < 3a/2$. These are stronger statements than those given in the problem.

Also solved by Jack Anderson and Orville Goering (jointly), P. M. Anselone, Leon Bankoff, A. J. Bosgang, W. B. Carver, Fred Discepoli, A. L. Epstein, Harry Furstenberg, Russell Godard and Vern Hoggatt and Robert Jamison (jointly), Douglas Holdridge, A. R. Hyde, M. S. Klamkin, S. Leja, A. E. Livingston, D. C. B. Marsh, George Millman, E. A. Nordhaus, Margaret Olmsted, M. J. Pascual, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, O. E. Stanaitis, A. V. Sylwester, Chih-yi Wang, and the proposer.

Ringenberg pointed out that the triangle $a=1$, $b=1$, $c=2(\sqrt{2}-1)$ is an example of a triangle of the type considered in the problem. Epstein remarked that if $s^2 = nab$, $n > 1$, then $s < na$, $s < nb$, $c < (n-1)a$, $c < (n-1)b$; the proof given above for the case $n=2$ is easily generalized to cover this broader situation, and yields the strengthened conclusion that $s < (n+1)a/2$.

Scalenity of a Triangle

E 1069 [1953, 331]. *Proposed by Robert Buehler, Arthur Gregory, and J. R. Wilson, Sandia Corporation, Albuquerque, N. M.*

How un-isosceles can a triangle be?

Solution by W. B. Carver, Cornell University. We must first define some measure of, let us say, the *scalenity* of a triangle. Assuming that no two sides of the triangle are to be equal, we let the lengths of the sides be $a > b > c$. Any measure of scalenity should be homogeneous of degree zero in a, b, c, so that two similar triangles should be equally un-isosceles. As a fairly satisfactory definition we shall

say that the scalenity of the triangle is the smaller of the two ratios a/b and b/c. (The ratio a/c is of course greater than either a/b or b/c.)

Let $x=a/b>1$ and $y=b/c>1$. Then $a=bx$ and $c=b/y$, and we must have $bx<b+b/y$, or $(x-1)y<1$. It follows that either x or y must be less than $(\sqrt{5}+1)/2$, and the scalenity of the triangle must be less than this familiar geometric constant.

Suppose we take an arbitrary line segment BC as the side a of the triangle. Let the point D divide this segment in extreme and mean ratio, so that $BC/BD=BD/DC$. Then we take the vertex A close to D but not, if we are to have a triangle, on the line BC. As we take A closer and closer to D, the scalenity of the triangle increases and approaches the limit $(\sqrt{5}+1)/2$. Thus there is no *most* un-isosceles triangle, because if A does not coincide with D we can always take a new A closer to D and giving a more un-isosceles triangle.

Also solved by C. M. Ablow, A. S. Gregory, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, A. E. Nordhaus, C. S. Ogilvy, L. A. Ringenberg, and Azriel Rosenfeld.

Editorial Note. Other definitions of *scalenity* might have been used, such as: (1) the smaller of $(a-b)/(a+b+c)$ and $(b-c)/(a+b+c)$, (2) the smaller of $A-B$ and $B-C$, (3) the smaller of A/B and B/C.

The problem suggests further investigations, such as an extension to n-sided polygons and a consideration of the analogous problem in more general metric spaces.

Differential Equation of All Conics of Given Eccentricity

E 1070 [1953, 332]. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

Solution by O. E. Stanaitis, St. Olaf College. Let the directrix be the y-axis and let the coordinates of the focus be $(p, 0)$. Then we obtain easily the equation of the general conic of given eccentricity e in the form

$$(x - p)^2 + y^2 = e^2x^2.$$

Differentiation and elimination of the parameter p leads to the first order differential equation

$$(e^2x - yy')^2 = e^2x^2 - y^2.$$

Also solved by John Jones, Jr., M. J. Pascual, and the proposer.

Editorial Note. The proposer hoped someone might find a Cartesian differential equation satisfied by all conics in the plane and having given eccentricity e. Halphen has given such a beautifully compact form of a Cartesian differential equation of all conics in the plane that one wonders if a similar achievement cannot be attained for the problem in hand. A Cartesian differential equation

of all parabolas in the plane (the case where $e=1$) is

$$(y''^{-2/3})'' = 0.$$

The set of all conics in the plane having given eccentricity e is given by the four-parameter family

$$(x-a)^2+(y-b)^2 = e^2(x\cos\omega + y\sin\omega - p)^2,$$

where the point (a, b) and the line $x\cos\omega+y\sin\omega-p=0$ are associated focus and directrix. Elimination of the parameters a, b, ω, p by successive differentiation would lead to the desired fourth order differential equation.

The proposer, using methods developed in his paper, *The differential equation of a conic and its relation to the aberrancy*, this MONTHLY [1952, pp. 531–538], found an *intrinsic* differential equation satisfied by all conics of given eccentricity e. The other solutions, like that of Stanaitis, restricted the family of conics in one way or another.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4568. *Proposed by A. W. Walker, University of Toronto*

It can be shown (*e.g.* by taking y as independent variable) that if $y^5-5y^3+5y-5x=0$, then $(4-25x^2)y''-25xy'+y=0$ (*Mathematical Gazette*, vol. 29, 1945, p. 223). Any three distinct roots of the above quintic equation must therefore be linearly related. Investigate this algebraically, and derive the following factored form of the quintic (readily verified by expansion):

$$(y-y_1)(y-y_2)(y-Ay_1-Ay_2)(y+y_1+Ay_2)(y+Ay_1+y_2) = 0,$$

where $A^2-A-1=0$.

4569. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingingdale, N. Y.*

Find explicitly a function $f(x)$ such that $f\{f(x)\}$ is of the order of magnitude of e^x. (In other words, find a function intermediate between x^n and e^x.)

4570. *Proposed by G. A. Dirac, King's College, London, England*

(1) If $d \geqq k \geqq 2$, show that there exist regular connected k-chromatic graphs of degree d and of arbitrarily high order.

(2) If $k \geqq 4$, construct a k-chromatic graph which does not contain a complete k-graph as a subgraph, and in which the degree of every node except one is $k-1$.

(See also problem 4526 [1953, 336]).

4571. *Proposed by Robert Kissling, Student, University of California, Berkeley*

Suppose $a^p+b^p=c^p$ in relatively prime integers with p a prime greater than 3. If q is any prime dividing a^2+ab+b^2 but not $a+b-c$, show that $(q-1)/6p$ is an integer.

4572. *Proposed by Paul Erdös, University of Notre Dame*

Let b_k be any sequence of non-negative real numbers such that

$$\overline{\lim} \frac{1}{n} \sum_{k=1}^{n} b_k < \infty.$$

Denote by $f(n)$ the number of b_k's with $1 \leqq k \leqq n$ for which $b_k > 0$. If $\underline{\lim}\, f(n)/n = 0$, show that

$$\sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

is irrational.

SOLUTIONS

An Application of a Jacobi Identity

4501 [1952, 469]. *Proposed by C. D. Olds, San Jose State College, California*

Show that

$$\log (1.1010010001 \cdots) = \frac{1}{11} + \frac{1}{2}\,\frac{1}{101} + \frac{1}{3}\,\frac{1}{1001} + \cdots.$$

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y. From a well known identity we have

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = 1 + x + x^3 + x^6 + \cdots, \quad |x| < 1,$$

where the nth term on the right is $x^{n(n-1)/2}$. Taking natural logarithms we have

$$\log (1 + x + x^3 + x^6 + \cdots) = \sum_{n=1}^{\infty} (-1)^n \log (1 - x^n)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \sum_{p=1}^{\infty} \frac{x^{pn}}{p} \right\}$$

$$= \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{x^{pn}}{p} \right)$$

$$= \sum_{p=1}^{\infty} \frac{x^p}{p(1+x^p)},$$

where the operations are justified by the absolute convergence. The proposed formula results from placing $x=0.1$.

Also solved by Leonard Carlitz, F. J. Duarte, H. M. Feldman, N. J. Fine, Harry Furstenberg, A. M. Gleason, H. W. Gould, Vern Hoggatt, M. S. Klamkin, L. L. Pennisi, O. E. Stanaitis, and Chih-yi Wang.

Isomorphic Groups

4502 [1952, 469]. *Proposed by Karl Goldberg, National Bureau of Standards, Washington, D. C.*

Prove that two groups A and B are isomorphic if and only if there exist operations "$\cdot$" and "$*$" defined over the elements of the groups as follows:

$$a \cdot b \in A \qquad a \cdot (b * a') = (a \cdot b)a'$$

$$b * a \in B \qquad b * (a \cdot b') = (b * a)b'$$

where $a, a' \in A$; $b, b' \in B$.

Solution by A. M. Gleason, Harvard University. If ϕ is an isomorphism of A onto B, then we can define "$\cdot$" and "$*$" by

$$a \cdot b = a\phi^{-1}(b) \qquad b * a = b\phi(a),$$

and the required relations are easily verified.

Conversely, suppose we are given groups A and B with "$\cdot$" and "$*$" defined satisfying the given relations. Let e_a and e_b be the identity elements of A, B, respectively. From

$$(e_a \cdot b)(a \cdot e_b) = e_a \cdot (b * (a \cdot e_b)) = e_a \cdot ((b * a)e_b) = e_a \cdot (b * a) = (e_a \cdot b)a$$

we deduce that $a \cdot e_b = a$ for all $a \in A$. Hence

$$e_a \cdot (e_b * a) = (e_a \cdot e_b)a = e_a a = a.$$

This, together with the analogous formula $e_b * (e_a \cdot b) = b$ proves that $\phi: a \to e_b * a$ is a one to one mapping of A onto B whose inverse is $b \to e_a \cdot b$. It is an isomorphism of the groups because:

$$\phi(a_1)\phi(a_2) = (e_b * a_1)(e_b * a_2) = e_b * ((a_1 \cdot e_b)a_2) = e_b * (a_1 a_2) = \phi(a_1 a_2).$$

Also solved by S. K. Berberian, T. A. Brown, R. M. Conkling, C. Gugenheim, N. G. Gunderson, T. H. Haynes, Jr., B. C. Kenny, C. E. Lemke and

Raphael Miller, D. C. B. Marsh, Yoshiv Nakamura, F. D. Parker, W. M. Perel, A. Jones Rodriguez, W. A. Rutledge, P. M. Treuenfels, L. M. Weiner, J. R. Wesson, J. V. Whittaker, and the Proposer.

An Inequality Concerning Distances

4503 [1952, 554]. *Proposed by H. S. Shapiro, Bell Telephone Laboratories, Murray Hill, N. J.*

Given n points $P_1, \cdots, P_n$ in $-1 \leqq x \leqq 1$, let τ_k denote the product of the distances from P_k to the other points. Prove

$$\sum_{k=1}^{n} \frac{1}{\tau_k} \geqq 2^{n-2},$$

and equality is attained for suitable P_i.

Solution by the Proposer. Denote the points by $x_1, \cdots x_n$ and set

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Now let $T(x)$ denote the Tchebychef polynomial of order $n-1$

$$T(x) = 2^{n-2}x^{n-1} + \cdots, \quad T(\cos\theta) = \cos(n-1)\theta,$$

whence $|T(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. Then by Lagrange interpolation we can write

$$T(x) = \sum_{k=1}^{n} \frac{f(x)}{(x - x_k)f'(x_k)} \cdot T(x_k),$$

and on equating coefficients of x^{n-1} we get

$$2^{n-2} = \sum_{k=1}^{n} \frac{T(x_k)}{f'(x_k)}, \qquad 2^{n-2} \leqq \sum_{k=1}^{n} \frac{1}{|f'(x_k)|}.$$

But $|f'(x_k)|$ is precisely the number τ_k defined above.

It is easily verified that equality holds if and only if the P_i are the points where $T(x)$ attains its maximum,

$$x_i = \cos\frac{(i-1)\pi}{n-1}.$$

Also solved by Robert Breusch and Fritz Herzog.

Postulates for a Group

4504 [1952, 554]. *Proposed by Olga Taussky, National Bureau of Standards Washington, D. C.*

Prove that a set S which is closed under an associative composition law which satisfies the following three axioms is a group:

1. There exists an idempotent e such that $e^2 = e$.

2. Every element has at least one left inverse with respect to e.
3. Every element has at most one right inverse with respect to e.

Solution by D. O. Ellis, University of Florida. Let x be any element of S and let y be some left inverse of x with respect to e. Then $yx=e$ and $e=e^2=(yx)(yx)=y(xyx)$. Hence, by uniqueness of right inverses, $x=xyx=xe$. The result now follows from the well known theorem: *a semigroup with a right identity and left inverses with respect to it is a group.*

To finish the proof independently of this theorem, let $zy=e$. Then $ex=zyx=ze=z$. Also, $exy=zy=e$. But, $e^2=e$, and so $xy=e$. Hence, $ex=xyx=xe=x$ and we have shown that e is an identity element and that left inverses are also right inverses, which conforms to the classic definition of group.

Also solved by Edward Assmus, Felice D. Bateman, T. A. Brown, M.O' N. Campbell, G. E. Collins, R. M. Conkling, F. E. Cothran, Kathryn Ellis, N. J. Fine, D. T. Finkbeiner, W. T. Fishback, Harley Flanders, Henry Furstenberg, J. W. Gaddum, Harry Goheen, E. E. Grace, P. S. Herwitz, J. Horváth, J. M. Hurt, Jack Indritz, P. W. M. John, Philip Johnson, B. C. Kenny, C. E. Lemke and R. Miller, T. C. Littlejohn, A. E. Livingston, D. C. B. Marsh, H. C. Miller, Jr., Kovina Milosevich, J. H. Oppenheim, F. D. Parker, R. F. Pavley, D. K. Pease, W. O. Portmann, G. B. Preston, D. W. Robinson, A. J. Rodriguez, Azriel Rosenfeld, W. A. Rutledge, D. W. Sasser, James Singer, W. A. Small, R. H. Sprague, F. W. Stallard, W. L. Stamey, D. R. Sudborough, G. H. M. Thomas, Peter Truenfels, T. A. Trumpler, L. M. Weiner, J. V. Whittaker, H. S. Wilf, J. E. Wilkins, Jr., A. B. Willcox and J. D. Baum, H. C. Wiser and A. T. Lundell, Jerrold Yos, and the Proposer.

A Trigonometric Series

4505 [1952, 554]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

1. Show that if $\theta_n=\pi/2\cdot 3^n$, then

$$\sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n \sin 2\theta_n}{\sin 3\theta_n}=\frac{1}{4}.$$

2. Show that if ϕ_p is the pth positive root of $\tan x=x$ and $\theta_n=\phi_p/3^n$, then

$$\sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n \sin 2\theta_n}{\sin 3\theta_n}=0.$$

Solution by Chih-yi Wang, University of Minnesota. Let S_N be the partial sum of N terms of the given series in both cases. Define $\theta_0=\pi/2$ in case 1, and $\theta_0=\phi_p$ in case 2. By using the identity

$$4\sin y \sin 2y/\sin 3y = \cot y - 3\cot 3y$$

we obtain, for all sequences $\{\theta_n\}$ where $\theta_{n+1}=\theta_n/3$,

$$S_N = \frac{1}{4}\sum_{n=1}^{N}\left[\theta_n \cot \theta_n - \theta_{n-1}\cot\theta_{n-1}\right] = \frac{1}{4}\left[\theta_N \cot\theta_N - \theta_0 \cot\theta_0\right].$$

This gives

$$\frac{1}{4}\theta_N \cot\theta_N, \qquad \frac{1}{4}\left[\theta_N \cot\theta_N - 1\right]$$

in the respective cases. Since, in both cases, $\theta_N \cot\theta_N \to 1$ as $N\to\infty$, the stated results follow.

Also solved by N. J. Fine, Harry Furstenberg, Calvin Foreman, O. E. Stanaitis, John Todd, Peter Ungar, and the Proposer.

Numbers Whose Factors Belong to a Given Sequence

4506 [1952, 554]. *Proposed by Paul Erdös, Notre Dame University*

Let $a_1<a_2<\cdots$ be an infinite sequence of real numbers, $\sum 1/a_i<\infty$. Denote by $n_0=1<n_1<n_2<\cdots$ the numbers of form $\prod a_i^{\alpha_i}$, where α_i are non-negative integers. Assume the number of $n_i\leqq x$ equals

$$\frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right)$$

where $f(x)$ is increasing and $f(x^2)/f(x)<c$, $f(2x)/f(x)\to 1$. Show that the number of $a_i\leqq x$ equals

$$\frac{x}{f(x)\sum_{i=0}^{\infty}\frac{1}{n_i}} + o\left(\frac{x}{f(x)}\right).$$

The conditions of the theorem are satisfied if, for example, $f(x)=(\log x)^c$, $c>1$.

Solution by the Proposer. Denote by $g(x)$ the number of $n_i\leqq x$. We assume that

$$(1) \qquad g(x) = \frac{x}{f(x)}, \qquad f(x) \text{ increasing}, \qquad \frac{f(2x)}{f(x)}\to 1.$$

$f(2x)/f(x)\to 1$ implies $f(x)=o(x^\epsilon)$, thus

$$(2) \qquad g(x) > x^{1-\epsilon}.$$

Consider

$$h_k(x) = g(x) - \sum_{i=1}^{k} g\left(\frac{x}{a_i}\right) + \sum_{\substack{1\leq i,j\leq k\\ i\neq j}} g\left(\frac{x}{a_i a_j}\right) + \cdots$$

$$+ (-1)^k g\left(\frac{x}{a_1 a_2 \cdots a_k}\right).$$

$h_k(x)$ clearly equals the number of integers $\leqq x$ composed of $a_{k+1}, a_{k+2}, \cdots$. Denote by $A(x)$ the number of a's $\leqq x$. Clearly $h_k(x)+k \geqq A(x)$, and it follows from $f(2x)/f(x) \to 1$ and $f(x) \uparrow$, that for any c, $f(cx)/f(x) \to 1$. Thus

$$g\left(\frac{x}{d}\right) = \frac{x}{d} \Big/ f\left(\frac{x}{d}\right) = (1 + o(1)) \frac{x}{df(x)}$$

and hence

$$h_k(x) = \left\{ \prod_{i=1}^{k} \left(1 - \frac{1}{a_i}\right) \right\} g(x) \cdot (1 + o(1)) = (1 + o(1)) \frac{x}{f(x)} \prod_{i=1}^{k} \left(1 - \frac{1}{a_i}\right).$$

Letting $k \to \infty$ we obtain from $h_k(x)+k \geqq A(x)$ and the convergence of $\sum 1/a_i$ that

$$A(x) \leqq (1 + o(1)) \frac{x}{f(x)} \prod_{i=1}^{\infty} \left(1 - \frac{1}{a_i}\right) = (1 + o(1)) \frac{x}{f(x) \sum_{i=0}^{\infty} \frac{1}{n_i}}. \tag{3}$$

On the other hand we have

$$A(x) > h_k(x) - \sum{}^{*} g\left(\frac{x}{a_i}\right) \tag{4}$$

where the $*$ indicates that the summation is extended over the a_i satisfying $a_k < a_i \leqq x^{1/2}$. (4) is clear since the left member is greater than the number of a's in the interval $(x^{1/2}, x)$, that is, all n's divisible by a smaller a have been omitted. Thus we have

$$A(x) > (1 + o(1)) \frac{x}{f(x) \sum_{i=0}^{\infty} \frac{1}{n_i}} - \sum{}^{*} g\left(\frac{x}{a_i}\right).$$

Now we only have to show that as $k \to \infty$

$$\sum{}^{*} g\left(\frac{x}{a_i}\right) = o\left(\frac{x}{f(x)}\right).$$

This estimate can be obtained as follows:

$$\sum{}^{*} g\left(\frac{x}{a_i}\right) = \sum_{a_k < a_i \leqq x^{1/2}} g\left(\frac{x}{a_i}\right) < \frac{1}{f(x^{1/2})} \sum_{a_k < a_i < x^{1/2}} \frac{x}{a_i}$$

$$< c \frac{x}{f(x)} \sum_{i=k+1}^{\infty} \frac{1}{a_i} = o\left(\frac{x}{f(x)}\right).$$

The attempt to dispense with the condition $f(x^2)/f(x) < c$ has not yet succeeded.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Fundamentals of College Mathematics. By R. E. Johnson, N. H. McCoy and Anne F. O'Neill. New York, Rinehart and Company, Inc., 1953. xiv+479 pages.

This text is designed for a first year course in college mathematics. It contains more than enough material for such a course. It is written with care and with rigor and contains an adequate number of exercises for a text in which ideas rather than techniques are stressed.

The main objective of the book is the introduction of the calculus. Chapters on analytic geometry and on algebraic topics are presented as they are needed for the study of the calculus. Chapter 1 discusses the number concept and some basic ideas of analytic geometry. Chapter 2 is devoted entirely to elementary trigonometry which may be omitted without destroying the continuity of the book. There is a short chapter (5) on mathematical induction which is followed by two chapters introducing the calculus. Chapter 8 is a study of polynomial functions and equations. Chapters 12 and 13, concerning conic sections and transformations of axes, interrupt the continuity of the treatment, since little or no use is made of the calculus in these chapters.

Proofs of theorems are given when feasible, otherwise it is clearly stated that they are assumed because the proofs are beyond the scope of the book. The proofs of some of the theorems are indicated as optional by being given in small print. The treatments of limit, continuity, and the definite integral as the limit of a sum are more thorough and rigorous than is usual in an elementary text. Despite the fact that many students will find difficulty with chapters 6 and 10 in which these concepts are presented, an experienced instructor should find pleasure in meeting this challenge. The student who uses this book as a text will not be required to revise his ideas concerning fundamental concepts as he continues his study of mathematics even though he does not fully grasp their significance and importance during this first experience with them.

The treatment of the integral calculus differs from that of the usual text in that the antiderivative is not introduced until after a complete discussion of the definite integral as the limit of a sum.

The authors are to be congratulated on writing an excellent book. It should be seriously considered for use as a text by those who are seeking to raise the level of their first year college mathematics course.

V. H. WELLS
Williams College

The Teaching of Secondary Mathematics. By C. H. Brown, Harper and Brothers, New York. Cloth, xi+388 pages, 1953. $4.00.

The principal novelty of this new and challenging book on the teaching of mathematics is its emphasis on the problem of the place of mathematics in education. Almost one-half of the book is devoted to this topic whereas in other books in the field the same topic takes up not more than from one-twentieth to one-seventh of the book.

This pre-occupation (almost obsession) of the author with philosophy makes the book more one on the philosophy of the teaching of mathematics than one on methods of teaching it. The author, of course, knows exactly what he is about, for he declares in his preface: "The function of a book on the teaching of mathematics is, therefore, to provide an analysis of educational theory as applied to the teaching of secondary mathematics; to present the considerations which should govern choice of subject matter and of method; and to suggest possible methods of solving typical teaching problems." (p. x)

The analysis presented by the author is superbly done. It is at its best in the discussion of the great controversy between the doctrine of specific objectives and the program of the "progressives." Clearly, the partisans of neither school will like Professor Brown's conclusion that the controversy has served: " . . . to increase rather than diminish the confusion regarding the place and function of mathematics in secondary education." (p. 116)

Professor Brown's own proposed resolution of the conflict is that the function of subject matter in education is to impart information about the world and to stimulate reflection upon the problems presented by the world. Now mathematics has great potentialities for contributing powerfully to both of these objectives. But these are potentialities, not peculiar virtues inherent in the subject. The teacher's function is to realize these potentialities.

It is a consequence of the author's approach to his subject that the remainder of the book may be found lacking in those specific kinds of help which the new teacher, particularly, looks for in a book on methods. Here Professor Brown again anticipates this possible criticism, for he says: "This book is based on the assumption that teaching method is individual; that a teacher who has an adequate knowledge of the subject matter of his field, who knows the needs, aspirations, and potentialities of his students, who understands the nature of the learning process, and whose philosophy of education is appropriate to a democratic society will be able to devise methods and techniques suited to his particular situation." (p. x) (It may be suggested parenthetically that these are no small requirements for teaching. They are, of course, the necessary and sufficient conditions for creative teaching.)

The principles for the guidance of the teacher in selecting subject matter and in devising methods are then presented in a chapter on demonstrative geometry, two chapters on the ninth year (ninth-year algebra and ninth-year general mathematics), and one on the other courses in the normal secondary

program in secondary mathematics. Chapters on general methods, evaluation, and the professional preparation of teachers complete the book.

"The Teaching of Secondary Mathematics" is a valuable addition to the professional library of any teacher of mathematics and it should prove to be a stimulating text-book for teachers in training. The young teacher, however, may find it necessary and advisable to supplement it with a more traditional text on methods. It is noteworthy that in its emphasis on understandings and concepts in the field of mathematical education the book is in harmony with the similar contemporary stress on concepts, meanings and understandings in the teaching of the subject matter of mathematics itself.

J. H. HLAVATY
New York City

Mathematics of Investment. By P. R. Rider and C. H. Fischer. New York, Rinehart. 1951. viii+228 pages+Tables+Index. $5.00.

Mathematics of Finance. By L. L. Smail. New York, McGraw-Hill, 1953. x+189 pages+Appendix+Tables+Answers+Index. $4.50.

Fundamental Procedures of Financial Mathematics. By M. Rassweiler and I. Rassweiler. New York, Macmillan, 1952. vi+236 pages+Appendix +Answers+Index. $3.25.

The first two of these books handle the subject of investment mathematics in the traditional algebraic manner, while the third uses arithmetic computations only.

The usual topics of simple and compound interest and discount, amortization and sinking funds, depreciation, bonds, life annuities and insurance are clearly and completely presented by both Smail and Rider and Fischer. Many problems, with answers to the odd numbered ones, appear in each book, Smail giving answers in the back of the book and Rider and Fischer immediately after the problems.

Rider and Fischer depart from tradition in several respects in the arrangement of the tables. Five compound interest and annuity functions are listed on a page under a given interest rate. This has obvious advantages since problems using several functions usually involve only one interest rate. Tabular values of $s_{\overline{n}|}^{-1}$ are listed instead of the usual $a_{\overline{n}|}^{-1}$; the reasons given for this change are the presence of $s_{\overline{n}|}^{-1}$ in the general annuity formulas and the slight advantage in replacing the subtraction of the interest rate by addition in obtaining the reading for the other reciprocal annuity factor. Rider and Fischer also include lower than usual interest rates in the tables to comply with modern lower market rates. Smail's tables follow the usual pattern for the compound interest and annuity functions. Both books use the Commissioner's 1941 Standard Ordinary Mortality Table based on 2½% interest rate. Rider and Fischer supplement this with the 1937 Standard Annuity Table also based on 2½%, but

with male and female entries. Both texts have the customary six-place logarithms and seven-place logarithms of interest ratios.

Rider and Fischer have made as practical and teachable as possible the chapters on general annuities, bonds, life annuities and insurance. Smail has an appendix devoted to pertinent topics from algebra. In conclusion both texts are quite suitable for the standard course in investment mathematics.

The Rassweiler and Rassweiler text requires no algebraic preparation. The procedure for each topic is thus: a short paragraph on definitions and basic ideas, a well emphasized statement of the main problem, an outline of the method of solution, an example stated and solved with the steps numbered and named in columns parallel to the work, and finally a set of problems, with answers to the odd numbered ones appearing in the back of the book. For the weaker student and one without algebraic preparation this technique seems good as long as the problems fall into one of the type forms. The better student, however, would profit more from a course using a text based upon algebra.

The usual topics of investment mathematics are treated with the exception of depreciation and capitalized cost. In addition there are chapters on taxation (property, income, excise and sales), commission, pricing and profits, fire and automobile insurance. The Commissioner's 1941 Table is included, but for the work on compound interest and annuities certain the student must get his own table.

J. O. BLUMBERG
University of Pittsburgh

Demand Analysis. By Herman Wold in association with Lars Jureen. New York, John Wiley and Sons, Inc., 1953. xvi+358 pages. $7.00.

This book is concerned with the behaviour of a consumer (or some particular set of consumers) when faced with economic choices. The larger part of it is given to a discussion of statistical demand analysis. Typically, in this type of analysis, it is supposed that the quantity of a particular commodity which consumers buy is determined causally by the economic conditions in which they find themselves. For example, if q_0 is the amount of a particular commodity bought, a useful hypothesis might be that q_0 is determined by the relation

$$q_0 = f(i, p_0, p_1, \cdots, p_n, \epsilon) \tag{1}$$

where i is the consumers' disposable income, $p_0, p_1, \cdots, p_n$ are the prices of the commodity and other commodities which the consumer might consider, and ϵ is a variable intended to represent the haphazardness of behaviour at different times. Usually ϵ is taken to be a random variable. Very often (1) is supposed to be a linear relation between the logarithms of the variables:

$$\log q_0 = a \log i + \sum_{j=0}^{n} b_j \log p_j + \log \epsilon.$$

It is then the task of statistical demand analysis to determine $a, b_0, \cdots, b_n$ from available data. Thus the subject requires the use of much probability and statistical theory.

The present book is divided into five parts. Part I is largely non-mathematical. It gives a defense of the use of classical least squares techniques in demand analysis. In Part II is a fairly extensive mathematical treatment of the deterministic theory of choice. Part III gives a summary of the properties of stationary random processes, and Part IV a summary of some theory of regression. Part V contains empirical results obtained from Swedish data.

Particularly interesting is Part III. This gives a summary of results on stochastic processes, including some recent work, which it is useful to have in such an accessible form. Part IV contains some interesting material on regression. These two parts of the book are more or less self-contained, and will be useful to statisticians generally.

As an exposition of its subject, the book is rather uneven, however. Throughout, it would be heavy going for a reader who was not already fairly well acquainted with the various aspects of its subject matter, and with the controversy about them. Part I, which is supposed to serve as an introduction, contains a particular point of view of what is the appropriate statistical method to use in analysing demand data. Professor Wold's contribution to this controversy is, in fact, a very important one, especially where he emphasizes the use of causal chains in setting up models of an economic system. Even with such models, however, the interactions between some of the variables may be much more rapid than those between others, and this fact leads to statistical difficulties in the analysis of data. Although Wold mentions these, he seems to dismiss them much too lightly.

The treatment of the theory of preference fields given in Part II is excellent. One flaw however, is in the discussion of the integrability condition, which is sketchy and seems rather misleading. The whole subject is an interesting one for mathematicians, and is still developing.

Summing up, the book, presenting as it does Professor Wold's contributions and personal views, is an important contribution to the literature of econometrics and mathematical economics.

C. B. WINSTEN
Cowles Commission and Oxford University

Theorie der Geometrische Konstruktionen. By Ludwig Bieberbach. Verlag Birkhauser, Basel, Switzerland. 1952. vi+162 pages. Bound, Fr. 18.70, unbound, Fr. 15.60.

Part of the author's purpose as stated in his preface is to make known the implications of the theory of geometric constructions to all mathematics. In this he is successful, as the reader will find material from projective and analytic geometry, the theory of algebraic equations, the structure of algebraic numbers, and number theory. For one section, extensive use is made of function theory.

The first twelve sections of the book deal with ruler and compass constructions in various combinations and modifications. Included are the Poncelet-Steiner constructions, and the Mohr-Mascheroni theory. In general, all quadratic constructions can be treated with these instruments.

Sections 13–15 still deal with ruler and compass constructions so far as they apply to the trisection of angles and the construction of regular polygons. In general, an angle cannot be trisected, but it is not always appreciated that there is an infinite set of angles whose trisection yields to ruler and compass. The investigation of regular polygons leads to a study of the cyclotomic equations involved and finally to the conclusion that regular polygons can be constructed if the number of their vertices is a prime of the form 2^p+1, the product of any two such numbers, or the product of any of these by a power of two.

With Section 16 begins the study of instruments which solve construction problems leading to third and fourth degree equations. The insertion-ruler allows the insertion of a fixed segment between two lines or circles while the ruler passes through a fixed point or is tangent to a circle. Trisection becomes simple, the duplication of the cube is achieved, and the construction of a regular heptagon becomes possible. The relation of this instrument to the conchoid and the limacon serves for analytic background. The carpenter's square with or without marked segments solves these problems and, in fact, the solution of the general cubic equation.

From Section 21 on, much use is made of the two pointed compass, involving a trial and error process which seems a far cry from the strict Euclidean constructions. But the author assures the reader that it is common practice among draftsmen and that he intends to use it as a formal, mathematical operation. A whole new field is opened up dealing with conics. The final instrument (Section 25) is the most general of all, namely, a sheet of transparent material on which any configurations may be traced and which may then be moved freely over the drawing board. Certainly a non-classical instrument, but here is Hjelmslev's Theorem: *A transparent sheet marked with a curve, e.g., a pair of perpendicular lines, in connection with the two point compass is sufficient to solve any construction problem which leads to an algebraic equation of any degree.*

The remainder of the book deals with approximate constructions, constructions on a sphere, and also presents Gelfond's proof of the transcendence of π. There are a few annoying misprints in relation to the figures. To this reviewer, there seems to be a curious unevenness in the presentation, in that simple processes are spelled out letter by letter whereas more complicated trains of thought are left to the reader's mathematical experience. All in all, it is a fine book to be recommended to all interested in geometry.

E. S. HAMMOND
Bowdoin College

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

NEW MAGAZINE FOR HIGH SCHOOL MATHEMATICS STUDENTS

For many years secondary school teachers of mathematics have felt a need for a journal written especially for the high school student. The National Council of Teachers of Mathematics, in cooperation with the Mathematical Association of America, is attempting to meet this need by the publication of a new journal called *The Mathematics Student Journal.*

The Mathematics Student Journal will contain enrichment and recreational material not found in the ordinary text-books. Alert teachers of mathematics have long been aware of the unlimited amount of enriching and stimulating material available on an elementary level in mathematics, but they have found that the press of busy schedules makes it difficult to look up this material and prepare it in a form for presentation to students. The new journal will aid greatly in solving this problem. Moreover, the journal will encourage the gifted student to continue his study of mathematics and will describe the many excellent opportunities awaiting him in the mathematical profession. A special feature will be a problem department to which students may contribute both problems and solutions. Because of its nature the new journal will appeal also to many adults with an interest in mathematics.

The Mathematics Student Journal will be edited by H. D. Larsen of Albion College. He will be assisted by Frank B. Allen of Lyons Township High School and Junior College, La Grange, Illinois. A second associate editor is to be named by the Association. Readers of this MONTHLY are invited and urged to submit material for publication in the new journal.

The Mathematics Student Journal will be issued four times a year during the months of October, December, February, and April. The first issue will be distributed in February, 1954. The subscription price will be 20¢ per year or 15¢ per semester. However, mailing will be done only in bundles of five copies or more, since the low subscription price does not permit the mailing of individual copies. Teachers should obtain subscriptions for their students and submit them in a group, all orders in a group running for the same period of time and being mailed to the same address. Subscriptions should be addressed to: National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

Editorial Note. Because of his new duties as editor of the above journal, Professor Larsen is retiring as editor of the Clubs Department of this MONTHLY. Since many of the functions of this department will be assumed by the new journal, publication of the Clubs Department will be discontinued after this issue.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

COOPERATIVE GRADUATE SUMMER SESSIONS IN STATISTICS

Beginning in 1954, North Carolina State College, the University of Florida, Virginia Polytechnic Institute and the Southern Regional Education Board will sponsor cooperative Graduate Summer Sessions in Statistics. The summer sessions will be of particular interest to the following: (1) research and professional workers who want intensive instruction in basic statistical concepts and who wish to learn modern statistical methodology; (2) teachers of elementary statistical courses who want some formal training in modern statistics; (3) prospective candidates for graduate degrees in statistics; (4) graduate students in other fields who desire supporting work in statistics; and (5) professional statisticians who wish to keep informed of advanced specialized theory and methods.

The courses are arranged to enable the person to take consecutive work in successive summers. The summer work in statistics may be applied at any one of the cooperating institutions in partial fulfillment of the requirements for a Master's degree. The catalog requirements for the degree must be met at the degree-granting institutions. Each doctoral candidate should consult with the institution from which he desires to obtain the degree regarding the applicability of the summer courses in statistics.

The first session will be conducted at Virginia Polytechnic Institute, June 9–July 17, 1954. Professor Maurice Kendall of the University of London will give a course in multi-variate analysis; Dr. Ralph Comstock of North Carolina State College will give one in quantitative genetics. The staff of the Department of Statistics of Virginia Polytechnic Institute will offer courses in probability and inference, analysis of variance, statistical methods, engineering statistics, education statistics, rank order statistics and the theory of sequential methods.

Inquiries should be addressed to Boyd Harshbarger, Head, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia.

"MATHEMATICS IN PUBLIC HIGH SCHOOLS"

A new publication entitled *Mathematics in Public High Schools*, Office of Education Bulletin 1953, No. 5, has been prepared by Dr. K. E. Brown. In this pamphlet such questions as the following are discussed. Are enough pupils taking mathematics in high school so our national supply of scientists and engineers can continue to increase? Are there as many pupils enrolled in mathematics in the secondary public schools as there were ten years ago? What mathematics, if any, is required of the pupils in the public high schools? In addition, the publication gives data about field trips, length of class periods, size of classes, number of teachers, and other pertinent information about mathematics in grades seven

to twelve. The pamphlet may be obtained by writing to: Superintendent of Documents, United States Government Printing Office, Washington 25, D. C.; price 20 cents.

PERSONAL ITEMS

Professor H. E. Bray of Rice Institute represented the Association at the inauguration of President Logan Wilson of the University of Texas on October 29, 1953.

Professor Dora McFarland has received a University of Oklahoma Foundation Teaching Award. This award of $500 was made for "extraordinary excellence in student counseling and teaching of freshmen and sophomores."

Agricultural and Mechanical College of Texas announces the following: Mr. M. L. Coffman and Mr. R. E. Collins, formerly students at the College, have been appointed to instructorships; Instructor T. Y. Hicks has resigned and is now an applied scientist with International Business Machines Corporation.

Bowling Green State University reports: Mrs. Florence S. Ogg has been appointed Lecturer; Associate Professor W. F. Cornell is on leave of absence for the year 1953–54 and is attending Ohio State University.

At Carnegie Institute of Technology: Professor A. E. Heins has been awarded a John Simon Guggenheim Fellowship and is studying in Denmark; Dr. H. G. Cohen, formerly a research associate at Hebrew Institute of Technology, Haifa, Israel, has been appointed to an assistant professorship.

At The Catholic University of America: Dr. Benjamin Lepson, formerly a mathematician with the Office of Naval Research, Washington, D. C., has been appointed to an assistant professorship; Dr. N. A. Wiegmann, previously a research mathematician at the National Bureau of Standards, Washington, D. C., has been appointed Visiting Professor.

Central Michigan College of Education announces the following: Assistant Professors D. R. Sudborough and H. W. Zeoli have been promoted to associate professorships; Miss Nikoline A. Bye has been elected Vice-President of the Michigan Council of Teachers of Mathematics for the year 1953–54.

Florida State University makes the following announcements: Dr. B. F. Hadnot, formerly mathematics teacher at Eglin Air Force Base, Florida, Dr. R. N. Tompson, previously a graduate student at Brown University, and Mr. A. V. Fend, recently a graduate student at the University of Illinois, have been appointed to assistant professorships; Miss Josephine Story, previously a graduate student at the University, has been appointed to an instructorship; Instructor J. M. Plant has been promoted to an assistant professorship.

At Illinois Institute of Technology: A series of eight monthly lectures for high school science teachers has been planned for the year 1953–54 to keep teachers informed of latest developments in their fields; Professor Karl Menger has given one of these lectures entitled "Geometry and Scientific Thought"; Professor Gordon Pall will give a lecture, "The Role of Amateurs in Number-Theory."

Massachusetts Institute of Technology announces: Professor Warren Ambrose was on leave during the fall semester of 1953–54 at Tata Institute of Fundamental Research, Bombay, India; Professor Witold Hurewicz and Professor Raphael Salem were also on leave during the fall semester of 1953–54 on Guggenheim Fellowships; Dr. J. F. Nash, Jr., previously C. L. E. Moore instructor, has been promoted to an assistant professorship; Dr. D. L. Wallace, formerly a graduate student at Princeton University, has been appointed C. L. E. Moore instructor.

McGill University reports the following: Assistant Professor R. T. Sharp of the University of Alberta has been appointed to an assistant professorship; Mr. P. J. Lanfer and Mr. C. D. McKay have been appointed Lecturers; Professor W. L. G. Williams has retired.

At Washington Square College, New York University: Dr. W. M. Hirsch, previously lecturer at Columbia University, has been appointed to an assistant professorship; Research Associate Professor Bernard Friedman has been promoted to a research professorship; Assistant Professor David Gans is on leave during 1953–54 on a Ford Foundation Fellowship; Research Associate Professor Keller is on leave during 1953–54 and is Head of the Mathematics Branch, Office of Naval Research.

Princeton University announces the following: Professor Charles Ehresmann of the University of Strasbourg was Visiting Professor of Mathematics during the fall term; Associate Professor Richard Otter of Notre Dame University is Visiting Lecturer during 1953–54; Dr. Felix Haas of Lehigh University has been appointed Henry Burchard Fine Instructor; Dr. L. S. Shapley and Dr. J. L. Snell have been reappointed Fine Instructors for the year 1953–54; D. A. Buchsbaum, Columbia University, and R. H. Crowell, Herbert Forrester, George Hufford and R. J. Semple, formerly graduate students at the University, have been appointed to instructorships.

Rutgers University announces: Assistant Professor Albert Wootton of Champlain College has been appointed Lecturer; Mr. N. R. Stanley, previously mathematician with the Republic Aviation Corporation, has been appointed to an instructorship; Professor R. E. Luce has received a Ford Foundation Grant for research abroad.

Southwestern Louisiana Institute reports: Assistant Professor P. M. Tullier, Jr., of Loyola University, Louisiana, has been appointed to an associate professorship; Mr. R. E. Allan, formerly of Tulane University, and Miss Margaret M. LaSalle, previously of Francis T. Nicholls Junior College, Louisiana State University, have been appointed to assistant professorships.

St. John's College announces the promotion of Assistant Professor A. H. Sarno to an associate professorship and of Instructor E. J. Germino to an assistant professorship.

Stanford University reports the following: Assistant Professor G. E. Latta of the University of British Columbia has been appointed Acting Assistant

Professor; Professor George Polya has retired with the title of Professor Emeritus.

At Stevens Institute of Technology: Professor L. Z. Pollara of Sienna College has been appointed to an associate professorship; Mr. M. E. White, recently with the United States Air Force, has been appointed to an instructorship.

The U. S. Naval Academy announces the following: Assistant Professor O. M. Thomas has been promoted to an associate professorship; Professor J. B Eppes has retired; Professor L. M. Kells has retired with the title of Professor Emeritus.

University of Arizona makes the following announcements: Mr. C. E. Aull, previously a graduate assistant at the University of Oregon, Miss Miriam J. Russell, formerly a graduate student at George Peabody College for Teachers, and Assistant Professor A. H. Steinbrenner of the United States Naval Academy, have been appointed to instructorships; Professor H. B. Leonard has retired with the title of Professor Emeritus.

The University of British Columbia announces: Dr. R. J. Wisner has been appointed to an instructorship; Associate Professor D. C. Murdoch has been promoted to a professorship.

The University of Chicago announces the following appointments to instructorships: Dr. Maurice Auslander, previously at Columbia University, and Dr. Serge Lang, formerly at the Institute for Advanced Study.

University of Cincinnati reports the following: Research Associate Arno Jaeger of the University of Illinois has been appointed to an associate professorship; Assistant Professor J. L. Baker, Jr., and W. E. Restemeyer have been promoted to associate professorships; Professors C. N. Moore and E. S. Smith have retired with the title of Professor Emeritus; during the first semester of 1953–54, Dr. Konrad Knopp of the University of Tübingen and Dr. Wolfgang Jurkat were in residence at the University.

University of Georgia makes the following announcements: Assistant Professor M. K. Fort, Jr., of the University of Illinois has been appointed to an associate professorship; Mr. B. K. Youse of Memphis State College has been appointed to an instructorship; Professor G. B. Huff has been appointed Head of the Department of Mathematics.

The University of Hawaii announces: Mr. M. J. Vitousek of Stanford University has been appointed to an instructorship; Associate Professor Christopher Gregory, who is also Chairman of the Department of Mathematics, has been promoted to a professorship.

The University of Maryland announces the following: Dr. G. C. Cree of Washington University, Dr. Gertrude Ehrlich, University of Tennessee, and Dr. Sol Schwartzman have been appointed to instructorships; Mrs. Eleanor B. P. Spencer has been appointed to a part-time instructorship; Instructor G. L. Spencer has been promoted to an assistant professorship.

The University of Minnesota reports the following: Dr. J. M. Slye of the

University of Texas has been appointed to an instructorship; Assistant Professors M. D. Donsker and B. R. Gelbaum have been promoted to associate professorships.

University of Nebraska announces: Assistant Professor R. M. Kozelka of Tufts College has been appointed to an assistant professorship; Dr. R. A. Moore of Washington University has been appointed to an instructorship; Associate Professor H. B. Ribeiro has been promoted to a professorship; Assistant Professor Lulu L. Runge has retired with the title of Assistant Professor Emeritus.

At the University of New Hampshire: Miss Evelyn J. Jones, formerly a mathematical analyst for the National Security Agency, Washington, D. C., has been appointed Graduate Teaching Assistant; Instructor Frederic Cunningham has been promoted to an assistant professorship.

The University of Richmond announces the following appointments: Associate Professor J. W. Sawyer of the Atlanta Division, University of Georgia, to an associate professorship; Mrs. Marion J. Stokes to an instructorship.

University of South Carolina announces the promotions of Associate Professor J. D. Novak to a professorship and Assistant Professor T. H. Lee to an associate professorship.

University of Southern California makes the following announcements: Associate Professors R. S. Phillips and P. A. White have been promoted to professorships; Professor R. S. Phillips is on leave during the year 1953–54 on a Guggenheim Fellowship and is at Yale University; Professor Ernst Snapper has received a National Science Foundation Fellowship Award for the year 1953–54 and is on leave to study at Harvard University; Associate Professor Albert Whiteman is on leave for the year 1953–54 and is at the Institute for Advanced Study; Dr. Harry Gonshor, previously a graduate student at Harvard University, has been appointed Visiting Instructor.

Dean Emeritus Samuel Beatty has been appointed Chancellor of the University of Toronto.

Reverend Stanley J. Bezuszka, formerly an assistant at Brown University, has been appointed Chairman of the Department of Mathematics of Boston College.

Dr. B. H. Bissinger, formerly vice-president of George Gillis Shoe Corporation, Fitchburg, Massachusetts, has been appointed to an associate professorship at Lebanon Valley College.

Mrs. Shirley A. Blackett, previously with the Educational Testing Service, Princeton, New Jersey, has been appointed to an instructorship at Northeastern University.

Mr. B. L. Blankenship, who has been teaching at San Saba High School, Texas, is now Head of the Department of Mathematics of Ranger Junior College, Texas.

Mrs. Helen L. Brooks has been appointed Lecturer at the University of Toledo.

Dr. D. R. Clutterham is now Senior Aerophysics Engineer with Consolidated

Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. H. K. Crowder of Case Institute of Technology has been promoted to an assistant professorship.

Assistant Professor Mary H. Cummings of the University of Rhode Island has been promoted to an associate professorship.

Mr. G. M. Eddington, formerly dean of men at Tusculum College, is teaching at Santa Monica High School, California.

Assistant Professor Herta T. Freitag has been named Head of the Department of Mathematics of Hollins College.

Mr. Otha Fuller, Jr., of Claflin College has been appointed to an instructorship at Agricultural, Mechanical, and Normal College, Pine Bluff, Arkansas.

Assistant Professor D. C. Gerneth of Mississippi Southern College has accepted a position as Senior Aerophysics Engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. Isidore Goldman is now Laboratory Supervisor for Greer Hydraulics, Inc., Brooklyn, New York.

Mr. Arnold Grudin, previously a part-time instructor at the University of Colorado, has been appointed to an instructorship at Denison University.

Miss Helen M. Heater has been appointed to an associate professorship at Alma College.

Dr. R. T. Herbst has a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Assistant Professor I. R. Hershner of the University of North Carolina has been appointed Professor and Chairman of the Department of Mathematics of the University of Vermont.

Associate Professor R. C. Hildner of the University of New Mexico is now a research staff member of Sandia Corporation, Albuquerque, New Mexico.

Dr. T. C. Holyoke of Northwestern University has been promoted to an assistant professorship.

Dr. M. A. Hyman, who has been on leave from the Naval Ordnance Laboratory, White Oak, Maryland, has been awarded the degree of Doctor of Science, Magna Cum Laude, by the Technical University of Delft, Holland.

Mr. R. J. Kohlmeyer of Lafayette College has been appointed to an assistant professorship at Pratt Institute.

Assistant Professor R. D. Larsson of Clarkson College has been promoted to an associate professorship.

Mr. R. J. Lemelin, formerly at the Willow Run Research Center, University of Michigan, has accepted a position as an analyst in the Machine Computation Laboratory, Research Department of the United Aircraft Corporation, East Hartford, Connecticut.

Associate Professor C. B. Lindquist of the University of Minnesota, Duluth Branch, has been promoted to a professorship.

Associate Professor M. St. J. MacPhail of Carleton College, Ottawa, Canada, has been promoted to a professorship.

Assistant Professor Morris Morduchow of Polytechnic Institute of Brooklyn has been promoted to an associate professorship.

Dr. C. V. Newsom, Associate Commissioner for Higher Education of the State of New York, served as Administrator of the Industry-College Conference which was held at White Sulphur Springs, West Virginia, November 12–13, 1953.

Dr. M. H. Protter, previously a member of the Institute for Advanced Study, has been appointed to an assistant professorship at the University of California.

Dr. Mina S. Rees, formerly head of the Mathematics Section, Office of Naval Research, has been appointed Dean of Faculty of Hunter College.

Mr. C. A. Rogers, previously a graduate student at the University of Washington, has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Assistant Professor J. A. Schumaker of MacMurray College for Women has been appointed to an instructorship at Grinnell College.

Associate Professor H. M. Schwartz of the University of Arkansas has been promoted to the position of Professor of Physics.

Professor Guy Stevenson, who is also Head of the Department of Mathematics of the University of Louisville, has been appointed Dean of the Graduate School.

Dr. G. L. Thompson of Princeton University has been appointed to an assistant professorship at Dartmouth College.

Miss Jane Uhrhan of Butler University has been promoted to an assistant professorship.

Mr. J. F. Wampler of York College has been promoted to a professorship.

Assistant Professor J. G. Wendel of Louisiana State University has been promoted to an associate professorship.

Assistant Professor J. R. Wesson of Birmingham Southern College has been promoted to a professorship.

Dr. Clement Winston, formerly with the War Production Board, Washington, D. C., is engaged now as a business economist in the United States Department of Commerce, Washington, D. C.

Professor H. A. Wright, who has served as Head of the Department of Mathematics of Transylvania College, has retired.

Miss Lois J. Younger, previously a student at William Jewell College, is teaching at Haddonfield High School, New Jersey.

The following have been appointed to teaching fellowships: University of Michigan, G. A. Paxson; University of Buffalo, A. H. Blessing, Miriam H. Brown, C. L. Gape, G. E. Neu, E. P. Rozycki, Joseph Siciliano.

The following have been appointed to graduate assistantships: Queen's University, W. A. Beyer; California Institute of Technology, D. L. Elliott; University of Florida, D. E. Thoro; University of Illinois, R. P. C. Caldwell, W. D. James, and E. A. Newburg; Yeshiva University, I. N. Katz.

Dean J. B. Brandeberry of the University of Toledo died on September 23, 1953; he was a charter member of the Association.

Professor Emeritus J. G. Hardy of Williams College died on September 5, 1953.

Senior Professor R. C. Lamb (retired) of the United States Naval Academy, died on July 22, 1953.

Professor Emeritus H. F. MacNeish of Brooklyn College, who was serving as a visiting professor at the University of Miami, died on August 28, 1953; he was a charter member of the Association.

Professor J. C. C. McKinsey of the Department of Philosophy of Stanford University died on October 26, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 113 persons have been elected to membership by the Board of Governors on applications duly certified.

RUTH J. ABRAMS, B.A. (Chicago) Technical Aide, Bell Telephone Laboratories, Murray Hill, N. J.

C. D. AUCHARD, M.A. (Colorado S. C.) Supervisor, Kansas State Teachers College.

C. E. AULL, M.S. (Oregon) Instr., University of Arizona.

V. E. BACH, B.S. (Brooklyn) Student, Brooklyn College.

J. G. BARON, M.D. (Budapest) 856 Wolfram Street, Chicago 14, Ill.

JAMES BATES, B.A. (Wayne) Grad. Research Mathematician, Visibility Laboratory, Scripps Institute of Oceanography, San Diego, Calif.

J. S. BENDAT, Ph.D. (Southern California) Lecturer, University of Southern California; Research Engineer, Northrup Aircraft, Incorporated, Hawthorne, Calif.

W. J. BERGER, M.S. (Carnegie) Mathematical Assistant, Carnegie Institute of Technology.

REV. S. J. BEZUSZKA, S.J., Ph.D. (Brown) Chairman, Boston College.

J. F. BLANSCHE, Student, University of Toronto.

J. W. BRACE, Ph.D. (Cornell) Instr., University of Maryland.

WALTER BRENTON, M.E. (Cornell) General Superintendent, Portland General Electric Company, Ore.

SOL BRODER, Student, Brooklyn College.

T. F. BROPHY, B.A. (Champlain) Grad. Student, New York State College for Teachers, Albany, N. Y.

P. S. BULLEN, M.S. (Natal) Lecturer, University of Natal, Durban, South Africa.

J. R. BYRNE, Ph.D. (U. of Washington) Asst. Professor, San Jose State College.

DORIS E. CAHN, B.A. (Newcomb) Mathematician, Redstone Arsenal, Huntsville, Ala.

J. M. CALLOWAY, Ph.D. (Pennsylvania) Asst. Professor, Carleton College.

C. J. CILLAY, B.A. (Texas) Private, United States Army.

G. F. CLANTON, B.S. (Baylor) Asso. Professor, Baylor University.

DEMETRIOS COUNES, M.A. (Johns Hopkins) Tutor, City College of the City of New York.

H. A. DANGEL, M.E. (Cincinnati) Professor, University of Cincinnati.

W. F. DARSOW, Ph.D. (Chicago) Instr., DePaul University.

MRS. MATILDE C. DEMACAGNO, Prof. Math. (La Plata) Professor, Secondary Schools of Cuyo University, San Juan, Argentina.

ALBERT DERIN, M.S. (Chicago) Instr., Kansas State College.

J. S. DORROH, B.A. (Mississippi C.) Superintendent, Springhill High School, Eupora, Miss.

S. S. DRAEGER, M.A. (Texas) Instr., Pan American College.

A. L. DUQUETTE, M.A. (Columbia) Instr., St. John's University.

MARY L. EDWARDS, M.A. (Fisk) Instr., Lincoln University.

H. M. ELDRIDGE, M.A. (Columbia) Asso. Professor, Fayetteville State Teachers College, N. C.

T. G. EVANS, Student, Princeton University.

MANUEL FELICIANO-DODONOFF, M.A. (Columbia) Instr., Catholic University of Puerto Rico.

S. R. FILIPPONE, M.A. (Wisconsin) Lecturer, Southern Illinois University.

R. E. FULLERTON, Ph.D. (Yale) Asst. Professor, University of Wisconsin.

A. E. FULTON, M.S. (Georgia) Asst. Professor, Georgia Institute of Technology.

I. S. GÁL, Ph.D. (Budapest) Instr., Cornell University.

T. A. GALIB, B.A. (Brown) Mathematician, United States Naval Underwater Ordnance Station, Newport, R. I.

D. C. GERNETH, Ph.D. (Oklahoma A. & M.) Senior Aerophysics Engineer, Consolidated Vultee Aircraft Corporation, Fort Worth, Tex.

KARL GOLDBERG, M.A. (Columbia) Mathematician, National Bureau of Standards, Washington, D. C.

D. F. GORMAN, Student, St. John's Preparatory, Brooklyn, N. Y.

H. V. GOSLING, Ph.D. (McGill) Retired, 271 Macdonnell Street, Kingston, Ont., Canada.

R. L. GRAVES, Ph.D. (Harvard) Assistant Head Engineer, Standard Oil Company of Indiana, Whiting, Ind.

SIDNEY GROSS, 48 Lenox Road, Rockville Centre, N. Y.

W. D. GUTSHALL, B.A. (U.C.L.A.) Analyst, Telecomputing Corporation, Burbank, Calif.

F. C. HALL, M.A. (Columbia) Asst. Professor, Manhattan College.

J. R. HATCHER, M.S. (Howard U.) Asst. Professor, Fisk University.

T. J. HEAD, Student, University of Oklahoma.

C. M. HEBBERT, Ph.D. (Illinois) Professor, Polytechnic Institute of Brooklyn.

R. T. HERBST, Ph.D. (Duke) Mathematician, Applied Physics Laboratory, Johns Hopkins University.

I. R. HERSHNER, JR., Ph.D. (Harvard) Professor and Chairman, University of Vermont.

C. S. HERZ, Ph.D. (Princeton) Instr., Cornell University.

J. G. HOCKING, Ph.D. (Michigan) Instr., Michigan State College.

WALTER HOFFMAN, M.A. (Wayne) Instr., Wayne University.

W. A. HORNING, Ph.D. (California) Theoretical Physicist, Hanford Works, Richland, Wash.

T. E. HULL, Ph.D. (Toronto) Asst. Professor, University of British Columbia.

L. K. JACKSON, Ph.D. (U.C.L.A.) Asst. Professor, University of Nebraska.

C. A. JOHNSON, Ph.D. (Kansas) Asst. Prof., Missouri School of Mines and Metallurgy.

D. L. JOHNSON, Student, University of Washington.

MOLLIE A. JONES, B.A. (Buffalo) Head of Department, Kenmore Junior and Senior High School, N. Y.

MRS. CAROL R. KARP, M.A. (Michigan S. C.) Instr., New Mexico College of A. and M. A.

M. A. KIRCHBERG, Student, Eastern Illinois State College.

REV. C. F. KOEHLER, S.J., M.A. (Boston C.) Asst. Professor, Loyola College.

A. V. KOZAK, D.Ed. (Penna. State) Chairman of Mathematics; Director, Kalgometrics Research Project, Concord College.

MRS. EMMA D. KUNDEL, B.S. (Huron) Instr., Sisseton Public School, S. D.

ROLANDO LARA, B.S. (Oklahoma) Seismograph Service Corporation, New Orleans, La.

J. W. LAWSON, Ph.D. (Chicago) Professor and Chairman, University of Manitoba.

C. W. LEININGER, M.S. (Arizona) Grad. Student, Southern Methodist University.

REV. J. J. MACDONNELL, S.J., M.A. (Boston C.) Grad. Student, Catholic University.

B. L. MACKIN, M.S. (Oklahoma) Asso. Professor, East Central State College, Ada, Okla.

CHARLOTTE M. MAIN, M.A. (New York S.T.C.) Head of Department, Baldwin School, Bryn Mawr, Pa.

EUGENE MALEK, Student, University of California.

EMANUEL MEHR, M.A. (Brooklyn) Instr., Brooklyn College.

A. S. MELTZER, B.S. (Syracuse) Student, Syracuse University.

E. A. MICHAEL, Ph.D. (Chicago) Asst. Professor, University of Washington.

P. W. MIELKE, JR., B.A. (Minnesota) Grad. Student, University of Chicago.

J. D. MILLER, Student, Eastern Illinois State College.

A. J. MUCCINO, B.S. (Waynesburg) Chemist, Treasury Department, Bureau of the Mint, Washington, D. C.

J. D. MUNN, M.A. (Alabama) Instr., Mississippi Southern Colleg

J. B. MUSKAT, Student, Yale University.

R. A. NIEMANN, M.A. (Illinois) Chief, Computation Divisio.., Naval Proving Ground, Dahlgren, Va.

J. A. NOHEL, Ph.D. (M.I.T.) Asst. Professor, Georgia Institute of Technology.

M. C. PALMER, M.A. (Vanderbilt) Instr., Westminster College.

F. A. PHILLIPS, M.A. (Oklahoma) Mathematician, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

JOEL PITCAIRN, M.A. (Harvard) Asst. Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pa.

R. G. POHRER, M.A. (Duke) Grad. Assistant, Duke University.

W. T. PUTNEY, III, M.A. (Washington S. C.) United States Army.

JOHN J. QUINN, Student, St. John's University.

M. I. ROSE, B.A. (Transylvania) Grad. Assistant, University of Kentucky.

SAM ROSENBERG, B.S. (Oklahoma) Grad. Student, Harvard University.

J. U. RUSSELL, M.A. (Mississippi) Grad. Assistant, University of Illinois.

LEO SARIO, Ph.D. (Helsinki) Research Fellow, Harvard Univ.

P. H. SAVET, Ph.D. (Paris) Head, Research Laboratory, Arma Corporation, Brooklyn, N. Y.; Instr., Brooklyn Polytechnic Institute.

R. H. SCHULZ, Student, University of Texas.

GEORGE SEIFERT, Ph.D. (Cornell) Asst. Professor, University of Nebraska.

S. H. SESSKIN, Student, Hofstra College.

A. L. SHIELDS, Ph.D. (M.I.T.) Instr., Tulane University.

A. J. SILVERMAN, B.A. (Temple) Assistant Instr., University of Kansas.

MARIETTA SMITH, M.A. (Appalachian S.T.C.) Teacher, Mars Hill High School, N. C.

MORRIS SMITH, M.S. (Columbia) Junior High School, Brooklyn, N. Y.

A. H. STEINBRENNER, M.A. (Columbia) Instr., University of Arizona.

ERMEL STEPP, JR., Airman Second Class, United States Air Force.

LOUISE C. STOLLE, M.A. (Minnesota) Instr., Berea College.

D. P. STURGIS, B.A. (American) Student, American University.

G. L. THOMPSON, Ph.D. (Michigan) Asst. Professor, Dartmouth College.

A. L. TRITTER, B.A. (Chicago) Grad. Student, University of Chicago.

H. E. URBAN, Student, University of Buffalo.

C. D. VAN WART, Student, Carnegie Institute of Technology.

G. H. VOSPER, 18 La Bar Row, Bangor, Pa.

J. R. WAGNER, B.S. in Ed. (Bloomsburg S.T.C.) Instr., North-Mont Joint School, Turbotville, Pa.

MRS. ELLA R. WATTS, M.A. (Columbia) Instr., Claflin College.

J. E. WETZEL, Student, Purdue University.

J. A. WILKINSON, B.S. (Oklahoma) Meteorological Officer, U. S. A. F., Florida State University.

JAMES WOLFE, Ph.D. (Harvard) Asso. Professor, University of Utah.

EMPLOYMENT OPPORTUNITIES

Beginning with the February 1954, issue, the MONTHLY will devote this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA, University of San Francisco.

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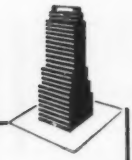

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